QUANTUM MECHANICS EXAM

18 January 2021

Answers sheet

We consider a one-dimensional system that is in a state whose wave function is given by

$$\psi(x) = \langle x|\psi\rangle = N e^{-\frac{\sigma}{2}x^2}\left(e^{ik_0x} + e^{-ik_0x}\right) = 2Ne^{-\frac{\sigma}{2}x^2}\cos(k_0x), \tag{1}$$

where N is a normalization constant

$$N = \left(2\sqrt{\frac{\pi}{\sigma}}\left(1 + e^{-k_0^2/\sigma}\right)\right)^{-1/2}. \tag{2}$$

(1) For the expectation value of x we find

$$\langle x\rangle = \int_{-\infty}^{\infty} dx\psi^*(x)x\psi(x) = 0, \tag{3}$$

because $\psi(x)$ is symmetric while x is anti-symmetric. Now for the expectation value of p we find

$$\langle p\rangle = \int_{-\infty}^{\infty} dx\psi^*(x)(-i\hbar)\frac{\partial\psi(x)}{\partial x} = 0, \tag{4}$$

where the integrand is also anti-symmetric.

(2) The probability distribution $\rho(x)$ for a measurement of position is

$$\rho(x) = |\psi(x)|^2 = 4N^2e^{-\sigma x^2}\cos^2(k_0x). \tag{5}$$

(3) The momentum wave function is found by calculating the Fourier transform of the position wave function:

$$\begin{aligned}
\psi(k) &= \int \frac{dx}{\sqrt{2\pi}} e^{ikx}\psi(x), &(6)\\
&= \int \frac{dx}{\sqrt{2\pi}} N e^{\frac{\sigma}{2}x^2}\left(e^{i(k-k_0)x} + e^{-i(k+k_0)x}\right), &(7)\\
&= \int \frac{dx}{\sqrt{2\pi}} N\left(e^{-\frac{\sigma}{2}(x+i\frac{1}{\sigma}(k-k_0))^2}e^{\frac{1}{\sigma}(k-k_0)^2} + e^{-\frac{\sigma}{2}(x+i\frac{1}{2\sigma}(k+k_0))^2}e^{\frac{1}{2\sigma}(k+k_0)^2}\right), &(8)\\
&= \frac{N}{\sqrt{\sigma}}\left(e^{-(k-k_0)^2/2\sigma} + e^{-(k+k_0)^2/2\sigma}\right), &(9)
\end{aligned}$$

where in the step from the third to the last line the Gaussian integral was performed for each of the two terms.

(4) The probability distribution $\rho(k)$ for a measurement of momentum is

$$\rho(k) = |\psi(k)|^2 = \frac{N^2}{\sigma}\left(e^{-(k-k_0)^2/\sigma} + e^{-(k+k_0)^2/\sigma} + 2e^{-(k^2+k_0^2)/\sigma}\right). \tag{10}$$

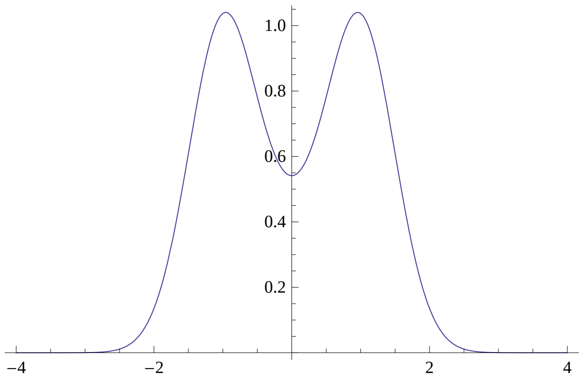

Figure 1: Qualitative shape of the momentum probability distribution $\rho(k)$ (the scale on the axes is arbitrary).

(5) The qualitative shape of the momentum probability distrbution is shown in Fig. 1: it is a superposition of two Gaussians centered in $\pm k_0$ respectively. Therefore, the most probable values of a measurement of momentum are at $k = \pm k_0$. The wave function Eq. (1) is a superposition of two eigenfuntions $\psi_\pm$,

$$\psi_\pm(x) = Ne^{\frac{-\sigma}{2}x^2} e^{\pm ik_0 x}. \tag{11}$$

These eigenfuctions correspond to two wave packets, both centered in the origin in position space, but moving in opposite directions along the x-axis, so centered about two opposite values $k = \pm k_0$ in momentum space.

(6) To find the expectation value of the energy, we have to evaluate $\langle H \rangle$.

Starting with (a)

$$\langle H \rangle = \langle \frac{p^2}{2m} \rangle = \frac{\hbar^2}{2m} \langle k^2 \rangle. \tag{12}$$

The simplest way of computing the integral is to use momentum space, in which case we get

$$\langle k^2 \rangle = \int_{-\infty}^{\infty} k^2 |\psi(k)|^2 dk \tag{13}$$

$$= \int_{-\infty}^{\infty} k^2 \rho(k), dk \tag{14}$$

where $\psi(k)$ is the momentum-space wave function Eq. (6) determined at point 3 and $\rho(k)$ is the momentum probability distribution Eq. (10) determined at point 4. All the integrals are of the form of the integrals found when computing the uncertainty in a wave packet. The result is

$$\langle H \rangle = \frac{\hbar^2}{2m} \langle k^2 \rangle = \frac{\hbar^2}{2m} \left(\frac{\sigma}{2} + \frac{k_0^2}{1 + e^{-k_0^2/\sigma}} \right). \tag{15}$$

For (b) we have

$$\langle H \rangle = \langle \frac{p^2}{2m} + \kappa x \rangle = \frac{\hbar^2}{2m} \langle k^2 \rangle + \kappa \langle x \rangle = \frac{\hbar^2}{2m} \langle k^2 \rangle, \tag{16}$$

where in the last step we have used Eq. (3). Therefore, the result is the same as for (a).

(7) Wave packets of minimal uncertainty are those for which

$$\Delta^2 x \Delta^2 p = \frac{\hbar}{4}. \tag{17}$$

The necessary and sufficient condition for a wave packet to have minimal uncertainty is

$$(\hat{p} - \langle\hat{p}\rangle)|\psi\rangle = i\lambda(\hat{x} - \langle\hat{x}\rangle)|\psi\rangle, \tag{18}$$

where $\lambda \in \mathbb{R}$.

The condition on the wave function is found computing

$$\langle x|(\hat{p} - \langle\hat{p}\rangle)|\psi\rangle = \langle x|\hat{p}|\psi\rangle = -i\hbar\partial_x\psi(x) = -i\hbar\left(\sigma x\psi(x) - 2Ne^{\frac{\sigma}{2}x^2}\sin(k_0x)k_0\right), \tag{19}$$

This is clearly not equal to the right hand side:

$$i\lambda\langle x|(\hat{x} - \langle\hat{x}\rangle)|\psi\rangle = i\lambda\langle x|(\hat{x}|\psi\rangle = x\psi(x). \tag{20}$$

Above we used the results of exercise 1, where we found that $\langle x\rangle = 0$ and $\langle p\rangle = 0$.

This means that the state of Eq. (1) does not satisfy the condition to be a state of minimal uncertainty. This is because Eq. (1) is not a wave packet, but a superposition of two wave packets.

(8) Assuming that the time-dependence is given by a free-particle Hamiltonian, the Heisenberg equations of motion for the operators x and p are

$$\frac{dx}{dt} = \frac{i}{\hbar}[H, x] = \frac{i}{\hbar}\frac{1}{2m}\left[p^2, x\right] = \frac{p}{m} \tag{21}$$

$$\frac{dp}{dt} = \frac{i}{\hbar}[H, p] = 0. \tag{22}$$

Using these equations we find that the time dependence of the position and momentum operators is

$$p(t) = p(0), \tag{23}$$

$$x(t) = x(0) + \frac{p(0)}{m}t. \tag{24}$$

The time dependence of the expectation values is then found to be

$$\langle p(t)\rangle = \langle p(0)\rangle = 0, \tag{25}$$

$$\langle x(t)\rangle = \langle x(0) + \frac{p(0)}{m}t\rangle = 0, \tag{26}$$

where we made use of the fact that $\langle p(0)\rangle = \langle x(0)\rangle = 0$, which are the results of exercise (1). Because time evolution is unitary, these results hold for all times.

(9) Before performing the measurement, the time evolution is the same as in the previous point, so for $t < 0$ Eqs. (25-26) are still true.

Upon performing the measurement, one finds a value for the momentum k with a probability given by Eq. (10). After this measurement, the wave function is a plane wave with momentum $p = \hbar k_m$, where k_m is the measured value.

Hence for $t > 0$ one has

$$\langle p(t)\rangle = k_m, \tag{27}$$

$$\langle x(t)\rangle = \langle x(0) + \frac{p(0)}{m}t\rangle = \langle x(0\rangle + \frac{\hbar k_m}{m}t, \tag{28}$$

where the mean value of position must be computed in a plane wave. In this state, the expectation value vanishes, but the uncertainty is infinite. So for all $t > 0$ the expectation value of position is ill-defined (indeterminate). (The answer found using $\langle x(t)\rangle = 0$ is also considered to be correct).

(10) For a free paticle the Hamiltonian is $H = p^2/2m$, with $p = \hbar k$. The time dependent wave function in momentum space is

$$\psi(k,t) = \langle k|\psi,t\rangle = \langle k|e^{-iHt/\hbar}|\psi\rangle = e^{-ik^2\hbar t/(2m)}\psi(k), \tag{29}$$

where $\psi(k)$ has been calculated in exercise 3. The time dependent wave function is used to determine the probability distribution of momentum at time t:

$$\rho(k,t) = |\psi(k,t)|^2 = |e^{-ik^2\hbar t/(2m)}\psi(k)|^2 = |\psi(k)|^2 = \frac{N^2}{\sigma}\left(e^{-(k-k_0)^2/\sigma} + e^{-(k+k_0)^2/\sigma} + 2e^{-(k^2+k_0^2)/\sigma}\right). \tag{30}$$

The result does not depend on time because momentum eigenstates the Hamiltonian is translationally invariant and momentum is conserved. Equivalently, momentum eigenstates are also energy eigenstates.

(11) In the limit $\sigma \to 0$, the probability distribution reduces to two Dirac Delta functions located at $k = \pm k_0$ (see Fig 4.1 and Eq. (4.23) of the textbook).

$$\lim_{\sigma\to 0}\rho(k,t) = \frac{1}{2}\left(\delta(k+k_0) + \delta(k-k_0)\right) \tag{31}$$