QUANTUM MECHANICS I EXAM

23 February 2021

Answers sheet

We consider a quantum system that can be found in four states, referred to as $|1\rangle$, $|2\rangle$, $|3\rangle$ and $|4\rangle$. Also consider the following system states:

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}[|1\rangle + (1+i)|2\rangle], \tag{1}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}[i|1\rangle + |3\rangle], \tag{2}$$

$$|\psi_3\rangle = \sqrt{\frac{2}{3}}|1\rangle + \sqrt{\frac{1}{3}}|4\rangle, \tag{3}$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}[|3\rangle + |4\rangle]. \tag{4}$$

(1) The probability P_i of a system prepared in state $|1\rangle$ to be detected in a state $|\psi_i\rangle$ is

$$P_i = |\langle 1|\psi_i\rangle|^2\,, \tag{5}$$

thus for the system states given by Eq. (1) to Eq. (4) above we have

$$P_1 = \frac{1}{3}, \qquad P_2 = \frac{1}{2}, \qquad P_3 = \frac{2}{3}, \qquad P_4 = 0. \tag{6}$$

(2) The normalization N is defined such that $|\langle\phi|\phi\rangle|^2 = 1$. Let us rewrite $|\phi\rangle$ by plugging in Eq. (1) to Eq. (4):

$$|\phi\rangle = N\left[|\psi_1 + 2i\sqrt{\frac{2}{3}}|\psi_2\rangle\right], \tag{7}$$

$$= \frac{N}{\sqrt{3}}\left[-|1\rangle + (1+i)|2\rangle + 2i|3\rangle\right]. \tag{8}$$

We now find

$$\langle\phi|\phi\rangle = \frac{N^2}{3}\left[\langle 1|1\rangle + 2\langle 2|2\rangle + 4\langle 3|3\rangle\right] = N^2\frac{7}{3}, \tag{9}$$

which means that the normalization constant N is

$$N = \sqrt{\frac{3}{7}}. \tag{10}$$

The probability that a system prepared in this state $|\phi\rangle$ will be detected in state $|1\rangle$ is

$$|\langle 1|\phi\rangle|^2 = \left|-\frac{N}{\sqrt{3}}\right|^2 = \frac{1}{7}. \tag{11}$$

(3) Now suppose that a system is in the state $|\phi\rangle$, a first measurement is performed which reveals that it is in the state $|\psi_2\rangle$. The probability of finding this result is

$$|\langle\psi_2 \mid \phi\rangle|^2 = N^2 \left|\langle\psi_2 \mid \psi_1\rangle + 2i\sqrt{\frac{2}{3}}\langle\psi_2 \mid \psi_2\rangle\right|^2, \tag{12}$$

$$= N^2 \left|-\frac{i}{\sqrt{2}\sqrt{3}}\langle 1 \mid 1\rangle + 2i\sqrt{\frac{2}{3}}\langle\psi_2 \mid \psi_2\rangle\right|^2 \tag{13}$$

$$= N^2\frac{3}{2} = \frac{9}{14}. \tag{14}$$

The probability that a subsequent measurement will reveal the system in a state $|1\rangle$ is

$$|\langle 1 \mid \psi_2\rangle|^2 = \left|\frac{i}{\sqrt{2}}\langle 1 \mid 1\rangle\right|^2 = \frac{1}{2}. \tag{15}$$

(4) An operator O_1 associated with an observable that takes the value $+1$ if the system is in the state $|\psi_2\rangle$, and 0 if the system is in any state orthogonal to $|\psi_2\rangle$ can be written in a basis of states $|1\rangle$, $|2\rangle$, $|3\rangle$ and $|4\rangle$ as

$$O_1 = |\psi_2\rangle\langle\psi_2| = \frac{1}{2}(i|1\rangle + |3\rangle)(-i\langle 1| + \langle 3|) = \frac{1}{2}\left(|1\rangle\langle 1| + i|1\rangle\langle 3| - i|3\rangle\langle 1| + |3\rangle\langle 3|\right), \tag{16}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{17}$$

(5) In exercise (4) is is stated that the operator takes the value $+1$ if the system is detected in the state $|\psi_2\rangle$ and 0 if the system is detected in any state orthogonal to $|\psi_2\rangle$. Thus the two eigenvalues of O_1 are $+1$ and 0, meaning the possible energy values of the system with a Hamiltonian $H = EO_1$ are E and 0.

The probability of an energy measurement for a system prepared in state $|\psi_1\rangle$ returning E is

$$|\langle\psi_2 \mid \psi_1\rangle|^2 = \left|\frac{1}{\sqrt{2}\sqrt{3}}(-i\langle 1| + \langle 3|)(|1\rangle + (1+i)|2\rangle)\right|^2 = \left|\frac{-i}{\sqrt{6}}\langle 1 \mid 1\rangle\right|^2 = \frac{1}{6}, \tag{18}$$

from this it follows that the probability of such a measurement returning an energy of 0 is 5/6.

(6) The basis of this system consists of four basis states, so a generic state vector depends on four complex constants and thus eight real parameters. However, because the normalization condition restricts one degree of freedom, we are left with seven free parameters. Then we are also only able to observe relative phases, and not the global phase, so the global phase is a non-observable parameter. Let us write a general vector of the given system as

$$|\varphi\rangle = Ne^{i\alpha}\left(|1\rangle + Ae^{i\beta}|2\rangle + Be^{i\gamma}|3\rangle + Ce^{i\delta}|4\rangle\right), \tag{19}$$

where N and α are the unobservable normalization and global phase we just mentioned. That leaves six observable parameters. For a system described by the Hamiltonian O_1, the eigenvalue 0 is trice degenerate and this means that another four parameters are indistinguishable when performing a measurement using the Hamiltonian, leaving two observable parameters. Let us now write the system as follows

$$|\varphi\rangle = Ne^{i\alpha}\left(|\psi_2\rangle + Ae^{i\beta}|\psi_{2,\perp}\rangle\right), \tag{20}$$

where $\psi_{2,\perp}$ represents a linear combination of states perpendicular to ψ_2. In the equation we now find the parameter A, and the relative phase β.

(7) The most general mixed density matrix ρ is the most general Hermitian operator with trace equal to one. The number of independent real components for a 2×2 Hermitian matrix is n^2: $\frac{1}{2}n(n-1)$ complex off-diagonal and n real diagonal elements. The total number of independent components for a general density matrix for a mixed state is thus $n^2 - 1$. In the case considered here $n = 4$ so the number of independent parameters is 15.

In the case of a pure state, the density matrix is a projector on a state, hence the number of independent parameters is equal to the number of observable parameters that characterize the state, determined ad the previous question, namely $2n - 2$, i.e. six in our case. This can equivalently determined as a number of idependent components for a hermitian matrix with a single nonvanishing eigenvalue, which is $2n - 1$, with the normalization condition providing and extra constraint, which gives again $2n - 2$.

(8) Two observables are compatible if their operators posses a common eigenbasis. In this case $|\psi_2\rangle$ is an eigenstate of O_1, but not of O_2. This can be seen by considering

$$O_2|\psi_2\rangle = |\psi_1\rangle\langle\psi_1 \mid \psi_2\rangle \neq \lambda|\psi_2\rangle, \tag{21}$$

where λ is a constant. Thus O_1 and O_2 are not compatible.

(9) The time dependent system $|1, t\rangle$, for which at $t = 0$ we have $|1, 0\rangle = |1\rangle$, is

$$|1, t\rangle = e^{-iHt/\hbar}|1\rangle. \tag{22}$$

Thus the probability of finding the system in state $|2\rangle$ at time $t = T$ is 0 since

$$\langle 1 \mid e^{iHT/\hbar} \mid 2\rangle = \langle 1 \mid 2\rangle = 0, \tag{23}$$

as a result of $|2\rangle$ being orthogonal to ψ_2 and thus corresponding to an eigenvalue of 0 of the Hamiltonian.

The probability of finding the system in state $|3\rangle$ at time $t = T$ on the other hand, does not vanish. Let us define a state orthogonal to ψ_2:

$$\psi_\perp = \frac{1}{\sqrt{2}}[-i|1\rangle + |3\rangle], \tag{24}$$

which allows us to rewrite $|3\rangle$ as

$$|3\rangle = \frac{1}{\sqrt{2}}\left[|\psi_2\rangle + |\psi_\perp\rangle\right]. \tag{25}$$

Using this we are able to determine the probability of finding the state $|3\rangle$ at time $t = T$:

$$\left|\langle 1 \mid e^{iHT/\hbar} \mid 3\rangle\right|^2 = \frac{1}{2}\left|\langle 1 \mid e^{iHT/\hbar} \mid \psi_2\rangle + \langle 1 \mid e^{iHT/\hbar} \mid \psi_\perp\rangle\right|^2, \tag{26}$$

$$= \frac{1}{2}\left|\langle 1 \mid e^{iET/\hbar} \mid \psi_2\rangle + \langle 1 \mid \psi_\perp\rangle\right|^2, \tag{27}$$

$$= \frac{1}{2}\left|\frac{i}{\sqrt{2}}e^{iET/\hbar}\langle 1 \mid 1\rangle - \frac{i}{\sqrt{2}}\langle 1 \mid 1\rangle\right|^2, \tag{28}$$

$$= \frac{1}{4}\left(e^{iET/\hbar 1} - 1\right)\left(e^{-iET/\hbar} - 1\right), \tag{29}$$

$$= \frac{1}{2}\left(1 - \cos\frac{ET}{\hbar}\right) = \sin^2\frac{ET}{2\hbar}. \tag{30}$$

(10) The operator associated with the observable O_3 can be written as

$$O_3 = |1\rangle\langle 1| - |3\rangle\langle 3|, \tag{31}$$

which we recognise as the Pauli matrix σ_3 (Eq. 3.133 from the lecture notes) in $|1\rangle$, $|3\rangle$ space. With O_3 time independent, the Heisenberg equation of motion in $|1\rangle$, $|3\rangle$ space reads

$$\frac{dO_3^H}{dt} = \frac{1}{i\hbar}[O_3^H, H^H] = \frac{E}{i\hbar}[O_3^H, O_1^H] = \frac{E}{2i\hbar}[O_3^H, \mathbb{I} - \sigma_2^H] = -\frac{E}{2i\hbar}[O_3^H, \sigma_2^H] = \frac{E}{\hbar}\sigma_1^H, \tag{32}$$

where σ_1 and σ_2 are the Pauli matrices given in eq. 3.131 and eq. 3.132 of the lecture notes and the superscript H makes explicit that the operators are in the Heisenberg picture.

(11) The operator O_3 in the Heisenberg picture is

$$O_3(t) = e^{iHt/\hbar} O_3 e^{-iHt/\hbar}, \tag{33}$$

where we can write the exponentiated Hamiltonian as

$$e^{iHt/\hbar} = e^{iEO_1 t/\hbar} = e^{iE(\mathbb{I}-\sigma_2)t/2\hbar} = e^{iEt/2\hbar} e^{-iE\sigma_2 t/2\hbar}. \tag{34}$$

that σ_2 is a Pauli matrix, and thus we have that $\sigma_2^2 = \mathbb{I}$, which we can use to work out the exponential:

$$e^{-iE\sigma_2 t/2\hbar} = \sum_{n\in\text{even}} \frac{(iEt/2\hbar)^n}{n!}\mathbb{I} - \sum_{n\in\text{odd}} \frac{(iEt/2\hbar)^n}{n!}\sigma_2, \tag{35}$$

$$= \cosh(iEt/2\hbar)\mathbb{I} - \sinh(iEt/2\hbar)\sigma_2. \tag{36}$$

The operator $O_3(t)$ is now found to be

$$O_3(t) = (c\mathbb{I} - is\sigma_2)\sigma_3(c\mathbb{I} + is\sigma_2) = c^2\sigma_3 + isc[\sigma_3, \sigma_2] + s^2\sigma_2\sigma_3\sigma_2 = c^2\sigma_3 + 2cs\sigma_1 - s^2\sigma_3, \tag{37}$$

where for convenience of notation we used $c = \cos(Et/2\hbar)$ and $s = \sin(Et/2\hbar)$.
Finally for we have that $O_3(t)$ in $|1\rangle$, $|3\rangle$ space in matrix form is

$$O_3(t) = \begin{pmatrix} \cos^2(Et/2\hbar) - \sin^2(Et/2\hbar) & 2\cos(Et/2\hbar)\sin(Et/2\hbar) \\ 2\cos(Et/2\hbar)\sin(Et/2\hbar) & -\cos^2(Et/2\hbar) + \sin^2(Et/2\hbar) \end{pmatrix}, \tag{38}$$

$$= \begin{pmatrix} \cos(Et/\hbar) & \sin(Et/\hbar) \\ \sin(Et/\hbar) & -\cos(Et/\hbar) \end{pmatrix}. \tag{39}$$

We can calculate the eigenvalues to find $\lambda_\pm = \pm 1$, with corresponding eigenvectors

$$|\psi\rangle_+ = \cos(Et/\hbar)|1\rangle + \sin(Et/\hbar)|3\rangle, \tag{40}$$

$$|\psi\rangle_- = \sin(Et/\hbar)|1\rangle - \cos(Et/\hbar)|3\rangle. \tag{41}$$

As an alternative approach to this problem, one could have started at equation Eq. (33) and used Baker-Campbell-Hausdorff, or write H in bra-ket notation to evaluate the expression at once, thereby omitting working out the product of the separate matrices.