QUANTUM MECHANICS I EXAM

18 June 2021

Answers sheet

We consider a one-dimensional system whose time evolution is given by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{1}$$

Suppose that at time $t = 0$ the system is in the state

$$|\psi\rangle = \sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|2\rangle, \tag{2}$$

where $|0\rangle$ and $|2\rangle$ are the ground state and the second excited state of H.

(1) Here we can use the of the annihilation operator a

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right), \tag{3}$$

as given in the hint, to write

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right). \tag{4}$$

Which we can in use to obtain the expected value of the position operator x:

$$\langle\psi|x|\psi\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\psi|\left(a + a^\dagger\right)|\psi\rangle = 0, \tag{5}$$

where we used

$$\left\langle m\left|a^\dagger\right|n\right\rangle = \delta_{m,n+1}\sqrt{n+1}, \tag{6}$$

$$\langle m|a|n\rangle = \delta_{m,n-1}\sqrt{n}. \tag{7}$$

(2) In similar fashion to what we did for the position operator x in Eq. (4), we can write the momentum operator p in terms of creation and annihilation operators:

$$p = -i\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^\dagger\right). \tag{8}$$

Again making use of the properties in Eq. (6) and Eq. (7), we can write

$$\langle\psi|p|\psi\rangle = -i\sqrt{\frac{m\omega\hbar}{2}}\langle\psi|\left(a - a^\dagger\right)|\psi\rangle = 0. \tag{9}$$

(3) The uncertainty of position in the state $|\psi\rangle$ is

$$\begin{aligned}
\langle\psi|(\Delta x)^2|\psi\rangle &= \langle\psi|x^2|\psi\rangle - \langle\psi|x|\psi\rangle^2, \\
&= \frac{\hbar}{2m\omega}\langle\psi|\left(a^2 + (a^\dagger)^2 + aa^\dagger + a^\dagger a\right)|\psi\rangle, \\
&= \frac{\hbar m\omega}{2}\frac{1}{3}\Big(\langle 0|\left(a^2 + (a^\dagger)^2 + aa^\dagger + a^\dagger a\right)|0\rangle \\
&\quad + 2\langle 2|\left(a^2 + (a^\dagger)^2 + aa^\dagger + a^\dagger a\right)|2\rangle \\
&\quad + \sqrt{2}\langle 0|\left(a^2 + (a^\dagger)^2 + aa^\dagger + a^\dagger a\right)|2\rangle \\
&\quad + \sqrt{2}\langle 2|\left(a^2 + (a^\dagger)^2 + aa^\dagger + a^\dagger a\right)|0\rangle\Big), \\
&= \frac{5\hbar}{2m\omega}
\end{aligned} \tag{10}$$

Where we used Eq. (4) and Eq. (5) in going from the first to the second line. And we used

$$a|n\rangle = \sqrt{n}|n-1\rangle \tag{11}$$
$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{12}$$

to go from the third to the final line.

(4) Similar to the previous exercise, we can calculate the uncertainty of momentum in the state $|\psi\rangle$:

$$\begin{aligned}\langle\psi|(\Delta p)^2|\psi\rangle &= \langle\psi|p^2|\psi\rangle - \langle\psi|p|\psi\rangle^2, \\ &= \frac{\hbar m\omega}{2}\langle\psi|\left(-a^2 - (a^\dagger)^2 + aa^\dagger + a^\dagger a\right)|\psi\rangle, \\ &= \frac{7\hbar m\omega}{6},\end{aligned} \tag{13}$$

where, to go from the first to the second line, we made use of Eq. (8) and Eq. (9), and then used Eq. (11) and Eq. (12) to obtain the final result.

By multiplying Eq. (10) and Eq. (13) we find

$$\langle\psi|(\Delta x)^2|\psi\rangle\langle\psi|(\Delta p)^2|\psi\rangle = \frac{15\hbar}{6m\omega}\frac{7\hbar m\omega}{6} = \frac{35}{12}\hbar^2. \tag{14}$$

This is larger than the minimal uncertainty $\frac{1}{4}\hbar^2$, which corresponds to a system in the ground state. Indeed, only Gaussian wave packets are minimum uncertainty state, but only the ground state of the harmonic oscillator is a Gaussian wave packet.

(5) Performing an energy measurement of the system at time $t = 0$ will make its wave function collapse into either of the two energy eigenstates $|0\rangle$ and $|2\rangle$ of which $|\psi\rangle$ is a superposition. The possible outcomes of this measurement are respectively E_0 with probability $\frac{1}{3}$ and E_2 with probability $\frac{2}{3}$. because the specturm of the harmonic oscillator is

$$E_n = \langle n|H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right), \tag{15}$$

the two values are $E_0 = \frac{1}{2}\hbar\omega$ and $E_2 = \hbar\omega\frac{5}{2}$.

Because energy is conserved, the probability does not depend on time. This can also be verified by explicit calculation: the time evolution of the state $|\psi\rangle$ as driven by the Hamiltonian H is

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi\rangle = \frac{1}{\sqrt{3}}e^{-\frac{i}{\hbar}E_0 t}|0\rangle + \sqrt{\frac{2}{3}}e^{-\frac{i}{\hbar}E_2 t}|2\rangle. \tag{16}$$

The probability of finding the system in $|0\rangle$ at time t is

$$P_0 = |\langle 0|\psi(t)\rangle|^2 = \left|\frac{1}{\sqrt{3}}e^{-\frac{i}{\hbar}E_0 t}\right|^2 = \frac{1}{3}, \tag{17}$$

and similarly the probability of finding the system in $|2\rangle$ at time t is

$$P_2 = |\langle 2|\psi(t)\rangle|^2 = \left|\sqrt{\frac{2}{3}}e^{-\frac{i}{\hbar}E_2 t}\right|^2 = \frac{2}{3}. \tag{18}$$

(6) For the Hamiltonian

$$H' = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(x + x_0\right)^2, \tag{19}$$

the Heisenberg equations of motion are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{i}{\hbar}[H, x] = \frac{i}{\hbar}\frac{1}{2m}\left[p^2, x\right] = \frac{p}{m}, \tag{20}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{i}{\hbar}[H, p] = \frac{i}{\hbar}\frac{1}{2}m\omega^2[(x + x_0)^2, p] = -m\omega^2(x + x_0). \tag{21}$$

(7) The solution is given in Eqs. (8.98 - 8.103) of the textbook.

(8) First, we need to know the time dependence of position $x(t)$ as governed by H'. In exercise (6) we found the equations of motion Eq. (20) and Eq. (21), which we can combine to find the second order differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -m\omega^2(x + x_0), \tag{22}$$

which we can in turn solve to find

$$x(t) = -x_0 + A\sin(\omega t) + B\cos(\omega t), \tag{23}$$

or since we are given initial conditions at $t = T$ it might be easier to write $x(t)$ in the form

$$x(t) = -x_0 + \alpha\sin(\omega(t - T)) + \beta\cos(\omega(t - T)). \tag{24}$$

To obtain the initial conditions of this differential equation, we determine the expected values of position and momentum in the fundamental state of the Hamiltonian H Eq. (1), for which we find

$$\langle 0|x|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 0|\left(a + a^\dagger\right)|0\rangle = 0, \tag{25}$$

$$\langle 0|\frac{\mathrm{d}x}{\mathrm{d}t}|0\rangle = \frac{1}{m}\langle 0|p|0\rangle = -i\sqrt{\frac{\omega\hbar}{2m}}\langle 0|\left(a - a^\dagger\right)|0\rangle = 0. \tag{26}$$

Finally, using these initial conditions, we find that the the expected value of the position at a time $t > T$ is given by

$$\langle 0|x(t)|0\rangle = -x_0 + x_0\cos(\omega(t - T)) \tag{27}$$

(9) Let us denote the potential of H as

$$V(x) = \frac{1}{2}m\omega^2x^2, \tag{28}$$

and the potential of H' as

$$V'(x) = \frac{1}{2}m\omega^2(x + x_0)^2. \tag{29}$$

We then note that the potential is given by

$$\langle x|V(\hat{x})|x'\rangle = V(x)\delta\left(x - x'\right), \tag{30}$$

and clearly

$$V'(x) = V(x + x_0). \tag{31}$$

Therefore

$$V'(x)\delta(x-x') = \langle x|V'(\hat{x})|x'\rangle = V(x+x_0)\delta(x-x') = \langle x+x_0|V(\hat{x})|x'+x_0\rangle = \langle x|T_{x_0}^{-1}V(\hat{x})T_{x_0}|x'\rangle, \tag{32}$$

meaning the potential $V'(\hat{x})$ can be obtained from $V(\hat{x})$ through a unitary transformation. This unitary transformation can be used to define the relation between H' and H as

$$H' = T_{x_0}^{-1} H T_{x_0}, \tag{33}$$

where T_{x_0} is the translation operator

$$T_{x_0} = \exp\left[-\frac{i}{\hbar} x_0 \hat{p}\right]. \tag{34}$$

Finally, we can use the unitarity of T_{x_0} to show that

$$T_{x_0} H' T_{x_0}^{-1} |n\rangle = H|n\rangle = E_n|n\rangle, \tag{35}$$

from which it immediately follows that

$$H' T_{x_0}^{-1} |n\rangle = E_n T_{x_0}^{-1} |n\rangle, \tag{36}$$

which means that the Hamiltonians H and H' have the same eigenvalues, and the relation between the eigenvectors of H and H' is

$$|n'\rangle = T_{x_0}^{-1} |n\rangle, \tag{37}$$

where $|n'\rangle$ represents the eigenvectors of H'.

(10) The action of transforming the momentum and position operators using Eq. (34) gives

$$T_{x_0}^{-1} x T_{x_0} = x + x_0, \tag{38}$$

$$T_{x_0}^{-1} p T_{x_0} = T_{x_0}^{-1} T_{x_0} p = p, \tag{39}$$

from which it follows that

$$a|n'\rangle = a T_{x_0} |n\rangle = T_{x_0} T_{x_0}^{-1} a T_{x_0} |n\rangle, \tag{40}$$

$$= T_{x_0} T_{x_0}^{-1} \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{i}{m\omega} p\right) T_{x_0} |n\rangle, \tag{41}$$

$$= T_{x_0} \left(a + x_0 \sqrt{\frac{m\omega}{2\hbar}}\right) |n\rangle, \tag{42}$$

$$= T_{x_0} \left(\sqrt{n}|n-1\rangle + x_0 \sqrt{\frac{m\omega}{2\hbar}} |n\rangle\right), \tag{43}$$

$$= \left(\sqrt{n}|(n-1)'\rangle + x_0 \sqrt{\frac{m\omega}{2\hbar}} |n'\rangle\right). \tag{44}$$

Thus it can be concluded that $|n'\rangle$ is an eigenstates of a for $n' = 0$, in which case the eigenvalue is $x_0 \sqrt{\frac{m\omega}{2\hbar}}$.

(11) The time dependent uncertainty of the position operator for a system prepared in the fundamental state of the Hamiltonian H Eq. (1) can be written as

$$(\Delta x(t))^2 = \langle 0| e^{-\frac{i}{\hbar} H' t} (x - \langle x \rangle)^2 e^{\frac{i}{\hbar} H' t} |0\rangle = \langle 0| (x(t) - \langle x(t) \rangle)^2 |0\rangle. \tag{45}$$

Since $x(t) - \langle x(t) \rangle$ is clearly invariant under translation we can write

$$(\Delta x(t))^2 = \langle 0| T_{x_0} e^{-\frac{i}{\hbar} H' t} (x - \langle x \rangle)^2 e^{\frac{i}{\hbar} H' t} T_{x_0}^{-1} |0\rangle, \tag{46}$$

$$= \langle 0| T_{x_0} e^{-\frac{i}{\hbar} E0 t} (x - \langle x \rangle)^2 e^{\frac{i}{\hbar} E_0 t} T_{x_0}^{-1} |0\rangle, \tag{47}$$

$$= \langle 0| T_{x_0} (x - \langle x \rangle)^2 T_{x_0}^{-1} |0\rangle, \tag{48}$$

$$\tag{49}$$

which shows that the uncertainty of position for this system does not depend on time.