**PROVA IN ITINERE DI FISICA QUANTISTICA**

20 luglio 2020

*Traccia di soluzione*

(1) Starting from the Hamiltonian

$$H_1 = \frac{p^2}{2m} + \lambda x \tag{1}$$

the Heisenberg equations of motion are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{i}{\hbar}[H, x] = \frac{i}{\hbar}\frac{1}{2m}\left[p^2, x\right] = \frac{p}{m} \tag{2}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{i}{\hbar}[H, p] = \frac{i}{\hbar}\lambda[x, p] = -\lambda, \tag{3}$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{i}{\hbar}[H, \lambda x] = \frac{i}{\hbar}\frac{\lambda}{2m}\left[p^2, x\right] = \frac{\lambda}{m}p \tag{4}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{i}{\hbar}[H, \frac{p^2}{2m}] = \frac{i}{\hbar}\frac{\lambda}{2m}[x, p^2] = -\frac{\lambda}{m}p. \tag{5}$$

Equation (3) shows that momentum is not conserved. This is due to the fact that the potential is not invariant upon space translations $x \to x + \delta$, i.e. the Hamiltonian does not commpute with $p$.

Equations (4-5) imply that $\frac{d}{dt}(T + V) = \frac{d}{dt}H = 0$, i.e. the energy is conserved. This is due to the fact that the potential is invariant upon time translations $t \to t + \delta$, so the hamiltonian commutes with the time-evolution operator.

(2) The time dependence of the position and momentum operators is

$$p(t) = p(t_0) - \lambda(t - t_0), \tag{6}$$

$$x(t) = x(t_0) - \frac{1}{2m}\lambda(t - t_0)^2 + \frac{p(t_0)}{m}(t - t_0), \tag{7}$$

where both (2) and (6) are used to obtain (7). The time dependence of the expectation values is thus

$$\langle p(t)\rangle = \langle p(0) - \lambda t\rangle = p_0 - \lambda t, \tag{8}$$

$$\langle x(t)\rangle = \langle x(0) - \frac{1}{2m}\lambda t^2 + \frac{p(0)}{m}t\rangle = x_0 - \frac{1}{2m}\lambda t^2 + \frac{p_0}{m}t. \tag{9}$$

The calculation of the uncertainties is also relatively straightforward, if a bit more laborious. For the uncertainty of the position we find

$$\Delta^2 x(t) = \langle x^2(t)\rangle - \langle x(t)\rangle^2 \tag{10}$$

$$= \langle x^2(0)\rangle - x_0^2 + \frac{t^2}{m^2}\left(\langle p^2(0)\rangle - p_0^2\right) + \frac{t}{m}\left(\langle x(0)p(0) + p(x)x(0)\rangle - 2x_0p_0\right) \tag{11}$$

$$= \Delta^2 x(0) + \frac{t^2}{m^2}\Delta^2 p(0) + \frac{t}{m}\left\langle \Delta x(0)\Delta p(0) + \Delta p(0)\Delta x(0)\right\rangle, \tag{12}$$

where to get from (10) to (11) equations (7) and (9) are plugged in (10) after which the squares are expanded and many of the terms cancel.

Since we are told to assume at $t = 0$ the special case of a Gaussian wave packet this expression can be further simplified, by recalling that the minimal uncertainty condition requires (recall Eqs. (6.53),(6.91) of the textbook)

$$\langle \Delta x(0)\Delta p(0) + \Delta p(0)\Delta x(0)\rangle = 0. \tag{13}$$

This can also be verified by explicit computation: assuming for simplicity and without loss of generality $x_0 = p_0 = 0$ we get

$$x(0)p(0) + p(0)x(0) = 2x(0)p(0) - i\hbar, \tag{14}$$

and

$$\langle\psi|\hat{x}\hat{p}|\psi\rangle = -i\hbar\sqrt{\frac{\alpha}{\pi}}\int_{\mathbb{R}} \mathrm{d}x x\,(-\alpha x)\,e^{-\alpha x^2} \tag{15}$$

$$= i\hbar\alpha\sqrt{\frac{\alpha}{\pi}}\int_{\mathbb{R}} \mathrm{d}x x^2 e^{-\alpha x^2} \tag{16}$$

$$= i\hbar/2 \tag{17}$$

so that indeed we get Eq. (13).

Using this result we thus get

$$\Delta^2 x(t) = \Delta^2 x(0) + \frac{t^2}{m^2}\Delta^2 p(0). \tag{18}$$

For the uncertainty of momentum we find

$$\Delta^2 p(t) = \langle p^2(t)\rangle - \langle p(t)\rangle^2 \tag{19}$$

$$= \langle p^2(0)\rangle - p_0^2 \tag{20}$$

$$= \Delta^2 p(0), \tag{21}$$

where similar to before we obtained (19) from (20) by plugging in equations (6) and (8) and expanding the squares, which leads to cancellation of the time dependent terms.

We can finally express the uncertainties Eqs. (24-20) in terms of $\sigma_0$, noting that for a minimum uncertainty state

$$\Delta^2 p(0) = \frac{\hbar^2}{4}\frac{1}{\Delta^2 x(0)} = \frac{\hbar^2}{4}\frac{1}{\sigma}. \tag{22}$$

We get finally

$$\Delta^2 p(t) = \frac{\hbar^2}{4}\frac{1}{\sigma}, \tag{23}$$

$$\Delta^2 x(t) = \Delta^2 x(0) + \frac{t^2}{m^2}\frac{\hbar^2}{4}\frac{1}{\sigma}. \tag{24}$$

(3) The eigenvalue equation is

$$\frac{\partial^2\psi(x)}{\partial x^2} = \frac{2m}{\hbar}\left(V(x) - E\right)\psi(x), \tag{25}$$

so the sign of $V - E$ plays an important role in determining the qualitative properties of the solutions as asked in this exercise.

The eigenfunctions of $H_1$ cannot be normalized in the proper sense. This is due to the fact that the potential satisfies $\lim_{x\to-\infty} V(x) = -\infty$ and as a result the energy is greater than the potential in this limit. The wave function therefore always has an oscillating trend in this limit (it isn't bound) and therefore the normalization integral diverges. The spectrum is consequently continuous as there are no constraints on $E$. The spectrum is however not degenerate. We can understand this as a consequence of the fact that $\lim_{x\to\infty} V(x) > E$ for each $E$. Therefore, in $x \to \infty$ the energy eigenfunction is exponential, but one of its exponential solutions diverges and is therefore not acceptable. Hence there is a unique solution for every $E$.

The potential $V_2 = \lambda|x|$ of the Hamiltonian $H_2$ is symmetric around $x = 0$ and has a unique minimum at this point. It furthermore satisfies $\lim_{x\to\pm\infty} V(x) > E$, meaning the eigenstates are bounded at both positive and negative infinity and hence are normalizable in the proper sense. This implies that the spectrum is not degenerate, given that in one dimension a normalizable spectrum is necessarily not degenerate. The spectrum is discrete as a consequence of the matching conditions at the origin.

(4) For the Hamiltonian $H_1$, we write the eigenvalue equation in the momentum basis as

$$\langle p|\hat{H}|\psi\rangle = E\langle p|\psi\rangle. \tag{26}$$

Recalling that $\langle p|\hat{x}|\psi\rangle = i\hbar(\partial/\partial p)\psi(p)$, we obtain

$$\frac{\partial}{\partial p}\psi_E(p) = -\frac{1}{i\hbar\lambda}\left(\frac{p^2}{2m} - E\right)\psi_E(p), \tag{27}$$

from which we can immediately find the solution to be

$$\psi_E(p) = \mathcal{N}\exp\left[\frac{i}{\hbar\lambda}\left(\frac{p^3}{6m} - Ep\right)\right], \tag{28}$$

where $\mathcal{N}$ is the normalization coefficient.

The solutions in position space can be written in terms of that in momentum space through a Fourier transform:

$$\psi_E^{(1)}(x) = \int \mathrm{d}p\langle x|p\rangle\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int \mathrm{d}p e^{ipx/\hbar}\psi_E(p). \tag{29}$$

For the Hamiltonian $H_2$, we divide the space in the two regions, region I, $x < 0$ and region II, $x > 0$. In region II the Hamiltonian $H_2 = H_1$ and the solution is given by

$$\psi^{II}(x) = \psi_E^{(1)}(x) \qquad x > 0. \tag{30}$$

with $\psi_E^{(1)}(x)$ given by Eq. (29). In region I, the hamiltonian is given by

$$H' = \frac{p^2}{2m} - \lambda x; \qquad x < 0. \tag{31}$$

It follows that in this region the wave function can be obtained by letting $\lambda \to -\lambda$ in Eqs.(28-30). It then immediately follows that

$$\psi^I(x) = k\psi^{II}(-x); \qquad x < 0. \tag{32}$$

where the coefficient of proportionality $k$ is determined by the matching conditions in the origin. Putting Eqs. (30-32) together we get that in this case the energy eigenfunction is given by

$$\psi_E^{(2)}(x)(x) = \psi_E^{(1)}(x)(|x|) \tag{33}$$

in terms of the energy eigenfunction $\psi_E^{(1)}(x)$ Eq. (29) of the Hamiltonian $H_1$. Note that now $E$ is quantized due to matching conditions at the origin, to be discussed at the next point.

(5) Concerning the Hamiltonian $H_1$, in exercise (3) it was discussed that the wave function corresponding to the Hamiltonian $H_1$ cannot be normalized in a proper sense. It is however possible to normalize the eigenfunctions in an improper sense by imposing $\langle\psi|\psi'\rangle = \delta(E - E')$:

$$\langle\psi|\psi'\rangle = |\mathcal{N}|^2 \int_{\mathbb{R}} \mathrm{d}p \exp\left[\frac{i}{\hbar\lambda}\left(E - E'\right)\right] = |\mathcal{N}|^2 2\pi\delta\left(\frac{E - E'}{\hbar\lambda}\right) = |\mathcal{N}|^2 2\pi\hbar\lambda\delta\left(E - E'\right), \tag{34}$$

from which we see that the normalization coefficient is

$$\mathcal{N} = \frac{1}{\sqrt{2\pi\hbar\lambda}}. \tag{35}$$

Concerning the Hamiltonian $H_2$, it should be noted that even though the potential has a discontinuous derivative in $x = 0$, the potential itself is continuous. This implies that the second derivative of the energy eigenfunctions is continuous in the origin, and thus the first derivative and the functions are also continuous. It follows that the matching conditions are

$$\psi^I(0) = \psi^{II}(0), \tag{36}$$

$$\psi^{I\prime}(0) = \psi^{II\prime}(0). \tag{37}$$

Note that these conditions can only be simultaneously satisfied together with Eqs. (28-29) by requiring that either $\psi^I(0) = \psi^{II}(0) = 0$, or $\psi^{I\prime}(0) = \psi^{II\prime}(0)$, so that we have that $k$ in Eq. (32) is respectively given by $k = -1$ (first case, anstisymmetric solutions) or $k = +1$ (second case, anstisymmetric solutions). These conditions, together with Eq. (28-29) , lead to the quantization of the values of $E$.