**QUANTUM MECHANICS EXAM**

23 September 2020

*Answers sheet*

We are given the Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \tag{1}$$

which describes a system with a harmonic potential. And for these exercises we are asked to consider the following states:

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}[|0\rangle + (1+i)|1\rangle], \tag{2}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}[i|0\rangle + |2\rangle], \tag{3}$$

$$|\psi_3\rangle = \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|3\rangle, \tag{4}$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}[|2\rangle + |3\rangle]. \tag{5}$$

(1) This example is discussed in chapter 8 of the lecture notes, where Eq. (8.49) represents the matrix $\langle m|\hat{x}|n\rangle$:

$$\langle m|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\delta_{m,n-1}\sqrt{n} + \delta_{m,n+1}\sqrt{n+1}\right), \tag{6}$$

which is also shown in Eq. (8.51) of the lecture notes.

Note in particular that because of the Kronecker deltas the only non-vanishing terms are those for which $m = n \pm 1$, from which we can immediately observe the following:

$$\langle\psi_2|\hat{x}|\psi_2\rangle = 0, \tag{7}$$

$$\langle\psi_3|\hat{x}|\psi_3\rangle = 0. \tag{8}$$

For the remaining two states we find

$$\langle\psi_1|\hat{x}|\psi_1\rangle = \frac{1}{3}\left[(1-i)\langle 1|\hat{x}|0\rangle + (1+i)\langle 0|\hat{x}|1\rangle\right] = \frac{2}{3}\sqrt{\frac{\hbar}{2m\omega}} \tag{9}$$

$$\langle\psi_4|\hat{x}|\psi_4\rangle = \frac{1}{2}\left[\langle 3|\hat{x}|2\rangle + \langle 2|\hat{x}|3\rangle\right] = \sqrt{\frac{3\hbar}{2m\omega}} \tag{10}$$

(2) Here we are given the state

$$|\phi\rangle = N\left[|\psi_1\rangle + 2i\sqrt{\frac{2}{3}}\,|\psi_2\rangle\right], \tag{11}$$

and asked to determine the normalization constant $N$ as well as the probability of finding $E = \frac{\hbar}{2}\omega$ upon measuring the energy.

To find the normalization constant we will have to solve

$$\langle\phi|\phi\rangle = 1. \tag{12}$$

First we express $|\phi\rangle$ in terms of basis vectors $|n\rangle$ :

$$|\phi\rangle = N\left[\left(\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}}\right)|0\rangle + \frac{1}{\sqrt{3}}(1+i)|1\rangle + i\frac{2}{\sqrt{3}}|2\rangle\right]. \tag{13}$$

Next we need use the fact that the states are orthonormal, meaning $\langle n|m\rangle = \delta_{n,m}$, which is given in Eqs. (8.42) of the lecture notes. Using this we find

$$\langle\phi|\phi\rangle = N^2\left[\frac{1}{3}\langle 0|0\rangle + \frac{2}{3}\langle 1|1\rangle + \frac{4}{3}\langle 2|2\rangle\right] = \frac{7}{3}N^2. \tag{14}$$

And thus we find that $N^2 = \frac{3}{7}$.

The energy eigenvalue $E_n$ is

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \tag{15}$$

which is given by Eq. (8.36) of the lecture notes. From this is can be concluded that the energy $E_0 = \frac{1}{2}\hbar\omega$ we are asked about, is the ground state energy. From Eq. (13) it can be concluded that the probability of a measurement of $|\phi\rangle$ resulting in a 'collapse' to the ground state is

$$P(E = \frac{1}{2}\hbar\omega) = P\left(|0\rangle \mid |\phi\rangle\right) = N^2\frac{1}{3} = \frac{1}{7}. \tag{16}$$

(3) First, we are asked to find the probability of a measurement on a system described by $|\phi\rangle$ revealing it is in the state $|\psi_2\rangle$. Using Eq. (11) we can see that this probability is

$$\begin{aligned} P\left(|\psi_2\rangle \mid |\phi\rangle\right) &= |\langle\psi_2|\phi\rangle|^2, && (17)\\ &= N^2\left|\langle\psi_2|\psi_1\rangle + 2i\sqrt{\frac{2}{3}}\langle\psi_2|\psi_2\rangle\right|^2, && (18)\\ &= N^2\left|-\frac{i}{\sqrt{6}}\langle 0|0\rangle + 2i\sqrt{\frac{2}{3}}\right|^2, && (19)\\ &= \frac{N^2}{3}\left|-\frac{1}{\sqrt{2}} + 2\sqrt{2}\right|^2, && (20)\\ &= \frac{3}{2}N^2 = \frac{9}{14}, && (21) \end{aligned}$$

where $N^2$ has been determined in the previous exercise.

Then we are asked in a follow up question what the probability is of a sequential energy measurement finding $E = \frac{\hbar}{2}\omega$, which is the energy corresponding to the ground state $|0\rangle$. The probability of finding this energy upon performing the second measurement is $P\left(|0\rangle \mid |\psi_2\rangle\right) = 1/2$, which can be found looking at Eq. (2).

(4) The time dependence of the position operator, where time transformation of the states is defined as

$$|n\rangle = e^{\frac{1}{i\hbar}H_0 t}|n\rangle, \tag{22}$$

is derived twice in the lecture notes, and the result is given in Eqs. (8.96,8.104):

$$\hat{x}(t) = \hat{x}(0)\cos(\omega t) + \frac{\hat{p}(0)}{m\omega}\sin(\omega t) \tag{23}$$

From we directly obtain the time dependence of the expectation value

$$\langle\hat{x}(t)\rangle = \langle\hat{x}(0)\rangle\cos(\omega t) + \frac{\langle\hat{p}(0)\rangle}{m\omega}\sin(\omega t). \tag{24}$$

Here Eq. (6) can be used to calculate the expectation values of position while Eq. (8.50) from the lecture notes can be used to calculate the expectation value of momentum:

$$\langle m|\hat{p}|n\rangle = -i\sqrt{\frac{m\hbar\omega}{2}}\left(\delta_{m,n-1}\sqrt{n} - \delta_{m,n+1}\sqrt{n+1}\right). \tag{25}$$

Thus we need to calculate $\langle\hat{x}(0)\rangle$ and $\langle\hat{p}(0)\rangle$ for both $|\psi_1\rangle$ and $|\psi_2\rangle$. Luckily for us, $\langle\hat{x}(0)\rangle$ has been calculated at exercise (1) so now we only have to calculate $\langle\hat{p}(0)\rangle$ in a similar way.

For $|\psi_1\rangle$ we find:

$$\begin{aligned}\langle\psi_1|\hat{p}(0)|\psi_1\rangle &= \frac{1}{3}\left[(1+i)\langle 0|\hat{p}(0)|1\rangle + (1-i)\langle 1|\hat{p}(0)|0\rangle\right] \\ &= \frac{2}{3}\sqrt{\frac{m\hbar\omega}{2}},\end{aligned} \tag{26}$$

and for $|\psi_2\rangle$ we find:

$$\langle\psi_2|\hat{p}(0)|\psi_2\rangle = 0. \tag{27}$$

Thus finally after plugging everything into Eq. (24), for the time dependence of the expectation value of position we find:

$$\langle\psi_1|\hat{x}(t)|\psi_1\rangle = \frac{2}{3}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t) + \frac{2}{3}\sqrt{\frac{\hbar}{2m\omega}}\sin(\omega t), \tag{28}$$

$$\langle\psi_2|\hat{x}(t)|\psi_2\rangle = 0. \tag{29}$$

(5) This new Hamiltonian now describes a free particle, and from Heisenberg´s equations of motion we get

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{i}{\hbar}[H, x] = \frac{i}{\hbar}\frac{1}{2m}\left[p^2, x\right] = \frac{p}{m}. \tag{30}$$

From which we find that the time dependence of the position operator is

$$x(t) = x\left(t_0\right) + \frac{p\left(t_0\right)}{m}\left(t - t_0\right). \tag{31}$$

From which we can find the time dependent expectation value at $t \geq 0$, for $t_0 = 0$:

$$\langle x(t)\rangle = \langle x\,(0)\rangle + \frac{\langle p\,(0)\rangle}{m}t, \tag{32}$$

where $\langle x\,(0)\rangle$ and $\langle p\,(0)\rangle$ have been determined in the previous exercise.

Collecting these results we find:

$$\langle x(t)\rangle = \frac{2}{3}\sqrt{\frac{\hbar}{2m\omega}} + \frac{2}{3}\sqrt{\frac{\hbar\omega}{2m}}t. \tag{33}$$

(6) We can determine $\varphi_2(x) = \langle x|2\rangle$ by relating it to $\varphi_0(x)$ using the creation operator and the relation:

$$|n\rangle = \frac{1}{\sqrt{n}}a^\dagger|n-1\rangle, \tag{34}$$

and the definition of the creation operator in terms of the position and momentum operators:

$$a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} - \frac{i\hat{p}}{m\omega} \right) \tag{35}$$

Namely, using we can write

$$\varphi_n(x) = \langle x|n\rangle = \frac{1}{\sqrt{n}} \sqrt{\frac{m\omega}{2\hbar}} \left( x - \frac{\hbar}{m\omega} \frac{\partial}{\partial x} \right) \varphi_{n-1}(x). \tag{36}$$

$\varphi_1(x)$ can be found in the lecture notes Eq. (8.65):

$$\varphi_1(x) = \sqrt{\frac{m\omega}{2\hbar}} 2x N_0 \exp\left(-\frac{m\omega x^2}{2\hbar}\right), \tag{37}$$

so we only have to calculate one step of this iterative series in Eq. (36)

$$\varphi_2(x) = \frac{1}{\sqrt{2}} \sqrt{\frac{m\omega}{2\hbar}} \left( x - \frac{\hbar}{m\omega} \frac{\partial}{\partial x} \right) \varphi_1(x), \tag{38}$$

$$= \frac{1}{\sqrt{2}} \frac{m\omega}{\hbar} N_0 \left( x - \frac{\hbar}{m\omega} \frac{\partial}{\partial x} \right) x \exp\left(-\frac{m\omega x^2}{2\hbar}\right), \tag{39}$$

$$= \frac{1}{\sqrt{2}} \frac{m\omega}{\hbar} N_0 \left( 2x^2 - \frac{\hbar}{m\omega} \right) \exp\left(-\frac{m\omega x^2}{2\hbar}\right), \tag{40}$$

$$= \frac{1}{\sqrt{2}} N_0 \left( 2\frac{m\omega}{\hbar} x^2 - 1 \right) \exp\left(-\frac{m\omega x^2}{2\hbar}\right) \tag{41}$$

(7) Here we are asked to determine the wave function

$$\chi(x) = \left\langle x \middle| \exp\left(\frac{i}{\hbar}\delta\hat{p}\right) \middle| \psi_2 \right\rangle. \tag{42}$$

$T_{-\delta} = exp\left(\frac{i}{\hbar}\delta\hat{p}\right)$ is the translation operator so we can write

$$\left\langle x \middle| \exp\left(\frac{i}{\hbar}\delta\hat{p}\right) \middle| \psi_2 \right\rangle = \langle x+\delta|\psi_2\rangle = \frac{1}{\sqrt{2}} \left( i\langle x+\delta|0\rangle + \langle x+\delta|2\rangle \right), \tag{43}$$

where $\langle x|0\rangle$ is given by Eq. (8.63) of the lecture notes:

$$\psi_0(x) = N_0 \exp\left(-\frac{x^2 m\omega}{2\hbar}\right), \tag{44}$$

and $\langle x|2\rangle$ has been determined in the previous exercise. Plugging those expressions into the equation above gives us

$$\chi(x) = \frac{N_0}{2} \left[ i\sqrt{2} + 2\frac{m\omega}{\hbar}(x+\delta)^2 - 1 \right] \exp\left(-\frac{(x+\delta)^2 m\omega}{2\hbar}\right) \tag{45}$$

(8) Here we are asked to find the uncertainty

$$\Delta^2 x = \langle x^2\rangle - \langle x\rangle^2, \tag{46}$$

at any time $t \geq 0$, for a system described by the Hamiltionian $H_0$ for all $t$, and which at $t = 0$ is in the state

$$|\bar{\chi}\rangle = \exp\left(\frac{i}{\hbar}\delta\hat{p}\right) |\, 0\rangle, \tag{47}$$

which describes a translation of the basis state $|0\rangle$. By transforming the position operator rather than the state vector it can be shown to have no effect on the uncertainty of the position operator:

$$\Delta^2 x = \langle \exp\left(\frac{-i}{\hbar}\delta\hat{p}\right) x^2 \exp\left(\frac{i}{\hbar}\delta\hat{p}\right)\rangle - \langle \exp\left(\frac{-i}{\hbar}\delta\hat{p}\right) x \exp\left(\frac{i}{\hbar}\delta\hat{p}\right)\rangle^2, \tag{48}$$

$$= \langle (x-\delta)^2\rangle - \langle (x-\delta)\rangle^2, \tag{49}$$

$$= \langle x^2\rangle - \langle x\rangle^2. \tag{50}$$

Here it was used that $\delta$ is a constant, which means that $\langle\delta\rangle = \delta$. From this it can be concluded that the uncertainty of the position operator is invariant under translation.

Because the translation does not affect the uncertainty, it can be concluded that the uncertainty can be expressed as $\Delta^2 x = \langle 0|x(t)^2|0\rangle - \langle 0|x(t)|0\rangle^2$. Which is obviously time independent as it consists of expectation values of an operator in an energy eigenstate. Thus the uncertainty at at any time is given by

$$\Delta^2 x = \langle 0|x^2|0\rangle - \langle 0|x|0\rangle^2, \tag{51}$$

where we have $\langle 0|x|0\rangle = 0$ from Eq. (6) and $\langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega}$ from

$$\left\langle n\left|\hat{x}^2\right|n\right\rangle = \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right), \tag{52}$$

which is Eq. 8.55 in the lecture notes.

Thus we finally obtain

$$\Delta^2 x = \frac{\hbar}{2m\omega}. \tag{53}$$