QUANTUM MECHANICS II EXAM

21 January 2021

Answers sheet

We consider a system of two particles in one dimension, whose dynamics are described by the Hamiltonian

$$H = \omega\left(a_1^\dagger a_1 + a_2^\dagger a_2\right) + \lambda\left(a_1^\dagger a_2 + a_2^\dagger a_1\right) + \mu \vec{s}_1 \cdot \vec{s}_2 = H_\omega + H_\lambda + H_\mu, \tag{1}$$

where the operators $a_i^\dagger$ and a_i satisfy the commutation relations

$$\left[a_i^\dagger, a_j\right] = \delta_{ij}; \quad \left[a_i^\dagger, a_j^\dagger\right] = [a_i, a_j] = 0. \tag{2}$$

(1) Here we consider the case where $\lambda = \mu = 0$, thus the Hamiltonian reduces to

$$H = H_\omega = \omega\left(a_1^\dagger a_1 + a_2^\dagger a_2\right). \tag{3}$$

The eigenvalues of H_i, with $i \in \{1, 2\}$ are found by noting that a_i, $a_i^\dagger$ satisfy the same commutation relations as the creation and annihilation operators for the one-dimensional harmonic oscillator. Therefore, $N_i = a_i^\dagger a_i$ is the number operator whise spectrum are the non-negative integers so

$$H_i|n_i\rangle = \omega a_i^\dagger a_i|n_i\rangle = \omega n_i|n_1\rangle, \tag{4}$$

and the eigenvalues are thus

$$E_i = \omega n_i. \tag{5}$$

From this it follows that the eigenvalues and aigenstates of the total Hamiltonian H_ω are

$$H_\omega|n_1, n_2\rangle = (H_1 + H_2)|n_1, n_2\rangle = \omega(n_1 + n_2)|n_1, n_2\rangle, \tag{6}$$

where we have defined

$$|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle. \tag{7}$$

The spectrum is thus

$$E_n^\omega = \omega(n_1 + n_2) = \omega n, \tag{8}$$

where $n = n_1 + n_2$.

(2) The degeneracy g for the case with spin-$\frac{1}{2}$ particles is $g = 4(n + 1)$, and for the spin-1 particles it is $g = 9(n + 1)$: $n + 1$ is the number of pairs of non-nehative integers whose sum is equal to n, while the factors 4 and 9 correspond to the spin degeneracies of the two particles. Indeed, for each spin-$\frac{1}{2}$ particle there are two spin states, and for each spin-1 there are three.

(3) Here we consider the Hamiltonian of Eq. (1) with $\lambda = 0$ but $\mu \neq 0$, for non-identical particles.

Let us consider

$$H_\mu = \mu \vec{s_1} \cdot \vec{s_2}, \tag{9}$$

which we can write in a basis of the eigenstates of the operators s^2, s_1^2 and s_2^2, where $\vec{s} = \vec{s_1} + \vec{s_2}$. In this basis the Hamiltonian becomes

$$H_\mu = \frac{\mu}{2}\left(s^2 - s_1^2 - s_2^2\right). \tag{10}$$

For spin-$\frac{1}{2}$ particles the eigenvalues are

$$E^{\mu}_{s} = \frac{\mu}{2}\hbar^2(s(s+1) - \frac{3}{2}), \tag{11}$$

where we can have $s = 0$ or $s = 1$, in which cases the eigenvalues are

$$E_{n,s=0} = E^{\omega}_{n} - \frac{3}{4}\hbar^2\mu, \tag{12}$$
$$E_{n,s=1} = E^{\omega}_{n} + \frac{1}{4}\hbar^2\mu. \tag{13}$$

For spin-1 particles the eigenvalues are respectively

$$E^{\mu}_{s} = \frac{\mu}{2}\hbar^2(s(s+1) - 4), \tag{14}$$

where in this case we can have $s = 0$, $s = 1$, or $s = 2$, which means the eigenvalues are

$$E_{n,s=0} = E^{\omega}_{n} - 2\hbar^2\mu, \tag{15}$$
$$E_{n,s=1} = E^{\omega}_{n} - \hbar^2\mu, \tag{16}$$
$$E_{n,s=2} = E^{\omega}_{n} + \hbar^2\mu. \tag{17}$$

(4) Here we remember the spin multiplicity $2s + 1$ discussed in exercise 1.

For the spin-$\frac{1}{2}$ case the degeneracies are

$g = n + 1$ for $E_{n,s=0}$,

$g = 3(n + 1)$ for $E_{n,s=1}$.

Therefore, the spin interaction splits the $4(n + 1)$ degenerate states correponding to each value of E^{ω}_{n} found in question (2) into two groups of $(1 + 3)(n + 1)$ states.

For the spin-1 case the degeneracies are

$g = n + 1$ for $E_{n,s=0}$,

$g = 3(n + 1)$ for $E_{n,s=1}$,

$g = 5(n + 1)$ for $E_{n,s=2}$.

Therefore, the spin interaction splits the $9(n + 1)$ degenerate states correponding to each value of E^{ω}_{n} found in question (2) into three group of $(1 + 3 + 5)(n + 1)$ states.

(5) Here we still consider the Hamiltonian of Eq. (1) with $\lambda = 0$ and $\mu \neq 0$, and we are given that $|\omega| \gg |\mu|\hbar^2$. Note that ω is assumed positive (otherwise theHamiltonian would notbhave a ground state), while μ is not necessarily positive. The condition means that the spacing of the energy levels due to spin is a fine structure on top of the main spacing due to the term proportional to ω. Now the particles are identical and thus the wave function must be symmetric for Bosons and antisymmetric for Fermions. For the cases below, the ket of the system is written in terms of the direct product kets Eq. (7). The spin state is included in the answer in order to keep track of (anti)-symmetry, but one could also have omitted it since it is not required for the exercise. We write the states as $|n, S\rangle$ where $n = n_1 + n_2$ and S is the value of the total spin.

First we consider the case of spin-$\frac{1}{2}$ particles, for which the wave function must be antisymmetric. Here the energy is given by

$$E_{n,s} = \omega n + \frac{\mu}{2}\hbar^2(s(s+1) - \frac{3}{2}). \tag{18}$$

The ground state is

$$|0,0\rangle = |0,0\rangle \otimes |0\rangle. \tag{19}$$

This state is non-degenerate. The ground state energy is

$$E_{0,0} = -\frac{3}{4}\hbar^2\mu. \tag{20}$$

If $\mu > 0$, the first excited state is

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|0,1\rangle + |1,0\rangle\right) \otimes |0\rangle. \tag{21}$$

The state is non-degenerate, and energy of this state is

$$E_{1,0} = \omega - \frac{3}{4}\hbar^2\mu. \tag{22}$$

The second excited state is

$$|1,1\rangle = \frac{1}{\sqrt{2}}\left(|0,1\rangle - |1,0\rangle\right) \otimes |1\rangle. \tag{23}$$

The degeneracy is 3, and energy of this state is

$$E_{1,1} = \omega + \frac{1}{4}\hbar^2\mu. \tag{24}$$

If $\mu < 0$ the ordering of these two states is reversed.

Now we consider the case of spin-1 particles, for which the wave function must be symmetric. Here the energy is given by

$$E_{n,s} = \omega n + \frac{\mu}{2}\hbar^2(s(s+1) - 4). \tag{25}$$

If $\mu > 0$, the ground state is

$$|0,0\rangle = |0,0\rangle \otimes |0\rangle, \tag{26}$$

where we have used the hint and noticed that when composing two spin 1 the $S = 0$ state is symmetric. This state is non-degenerate. The ground state energy is

$$E_{0,0} = -2\hbar^2\mu. \tag{27}$$

The first excited state is

$$|0,2\rangle = |0,0\rangle \otimes |2\rangle. \tag{28}$$

The degeneracy is 5. The energy is

$$E_{0,2} = \hbar^2\mu. \tag{29}$$

Again, if $\mu < 0$ the ordering of these two states is reversed.

The second excited state is

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|0,1\rangle + |1,0\rangle\right) \otimes |0\rangle. \tag{30}$$

This state is non-degenerate. The energy is

$$E_{1,0} = \omega - 2\hbar^2\mu. \tag{31}$$

Students who tacitly assumed $\mu > 0$ are given full score on this question, those who realized that the answer depends on the sign of μ get bonus points.

(6) Here we consider a composition of three spin-1 particles, where each particle can be in one of three possible spin states. This means that there are $3^3 = 27$ possible states for this composition of particles.

First, we can combine the first two particles after which we find a symmetric spin-2 state, an anti-symmetric spin-1 state and a symmetric spin-0 state:

$$|2,m\rangle_{12}\,|1,\{-1,0,1\}\rangle_3\,,\quad |1,m\rangle_{12}\,|1,\{-1,0,1\}\rangle_3\,,\quad |0,m\rangle_{12}\,|1,\{-1,0,1\}\rangle_3\,.$$

Now we can also include the third particle in the combination, obtaining the states $|s, s_z, s_{12}\rangle_{123}$:

$$|3,s_z,2\rangle_{123},\quad |2,s_z,2\rangle_{123},\quad |1,s_z,2\rangle_{123},\quad |2,s_z,1\rangle_{123},\quad |1,s_z,1\rangle_{123},\quad |0,s_z,1\rangle_{123},\quad |1,s_z,0\rangle_{123}.$$

Thus eigenvalues with $s = 3$ have a degeneracy of 7, eigenvalues with $s = 2$ have a degeneracy of $2 \times 5 = 10$, eigenvalues with $s = 1$ have a degeneracy of $3 \times 3 = 9$, and the eigenvalue with $s = 0$ is non-degenerate.

(7) The first excited level of the unperturbed Hamiltonian is twice degenerate, corresponding to the two eigenstates Eq. (7) $|10\rangle$ and $|01\rangle$. In order to determine the perturbative corrections we compute the expectation value of the perturbation in the degenerate subspace. We get (writing the bras as $\langle n_1 n_2|$)

$$\langle H_\lambda\rangle = \begin{pmatrix} \langle 10|H_\lambda|10\rangle & \langle 10|H_\lambda|01\rangle \\ \langle 01|H_\lambda|10\rangle & \langle 01|H_\lambda|01\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \lambda\sigma_1 \tag{32}$$

where in the last step we have recognized the Pauli matrix σ_1:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{33}$$

Now let us diagonalize the matrix $\lambda\sigma_1$, such that $D = M^{-1}\sigma_1 M$, where D is a diagonal matrix. In doing so find the eigenvalues and corresponding eigenvectors. We find easily that the eigenvalues of σ_1 are ± 1 and the diagonalization matrix is the orthogonal (unitary, real) matrix

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{34}$$

which can be used to transform the basis of the operators or the eigenstates.

The eigenvectors of the perturbation are

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle \pm |0,1\rangle\right). \tag{35}$$

and the eigenvalues are $\pm\lambda$. Therefore, the correction to the eigenvalue of the first excited order is

$$E^\lambda_{1,\pm} = \langle\pm|H_\lambda|\pm\rangle = \lambda\langle\pm|a_1^\dagger a_2 + a_2^\dagger a_1|\pm\rangle = \pm\lambda. \tag{36}$$

Taking into account this correction, the first excited state is no longer degenerate.

(8) Here we recognise that the contriobution proportional to λ to the Hamiltonian can be written as

$$H_\lambda = \lambda \vec{a}^\dagger \sigma_1 \vec{a}, \tag{37}$$

where again σ_1 is the Pauli matrix. We further note that upon a unitary transformation M of the operators a the term proportional to ω in the Hamiltinoian H is unchanged, because

$$H_\omega = \omega a_i^\dagger a_i = \omega a_i^\dagger M^\dagger_{ii'} M_{i'i} a_j. \tag{38}$$

It follows that if we define

$$b_i = M_{ij} a_j, \tag{39}$$

and we choose M as the unitary matrix M Eq. (34) which diagonalizes the Pauli matrix σ_1 we find

$$H_\lambda = \lambda \vec{b}^\dagger M \sigma_1 M^\dagger \vec{b} = \lambda \left(b_1^\dagger b_1 - b_2^\dagger b_2 \right) \tag{40}$$

and thus

$$H = \omega \left(b_1^\dagger b_1 + b_2^\dagger b_2 \right) + \lambda \left(b_1^\dagger b_1 - b_2^\dagger b_2 \right), \tag{41}$$

with the spectrum

$$E_{n_1,n_2} = \omega(n_1 + n_2) + \lambda(n_1 - n_2). \tag{42}$$

which shows that the first order perturbative result for the first excited state is the same as the exact result.

(9) Here we again consider $H = H_\omega + H_\lambda$. The ground state of this Hamiltonian is annihilated by both the operators b_i. But these are the linear combination Eq. (39) of the operators a_i, so this is the same as the ground state of the hamiltonian H_ω. It follows that at time $t = 0$ the system is in the state

$$a_1^\dagger |00\rangle = |10\rangle. \tag{43}$$

This state can be written as a linear combination of the two states $|\pm\rangle$ Eq. (35), which are eigenstates of the full hamiltonian H:

$$|1,0\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle + |-\rangle \right). \tag{44}$$

We thus get that the state of the system at time t is

$$|\psi, t\rangle = e^{-\frac{i}{\hbar} H t} |1,0\rangle, \tag{45}$$

$$= e^{-\frac{i}{\hbar}(\omega \mathbb{I} + \lambda \sigma_1) t} \frac{1}{\sqrt{2}} \left(|+\rangle + |-\rangle \right), \tag{46}$$

$$= e^{-\frac{i}{\hbar} \omega t} \frac{1}{\sqrt{2}} \left(e^{-\frac{i}{\hbar} \lambda t} |+\rangle + e^{\frac{i}{\hbar} \lambda t} |-\rangle \right), \tag{47}$$

$$= e^{-\frac{i}{\hbar} \omega t} \left(\cos\left(\frac{\lambda}{\hbar} t \right) |1,0\rangle - i \sin\left(\frac{\lambda}{\hbar} t \right) |0,1\rangle \right). \tag{48}$$

Thus the probability of finding the system in a state $|\phi\rangle$ at time $t = T$ is

$$P = |\langle \phi | \psi, T \rangle|^2 = |\langle 1,0 | \psi, T \rangle|^2 = \sin^2 \left(\frac{\lambda}{\hbar} T \right). \tag{49}$$