QUANTUM MECHANICS II EXAM

23 June 2021

Answers sheet

Consider a three-dimensional system whose dynamics is described by the Hamiltonian

$$H = \frac{(\vec{p} - \vec{A}(\vec{x}))^2}{2m} - \frac{e^2}{r}, \tag{1}$$

where $\vec{x}$ and $\vec{p}$ are the position and momentum operators, $r = |\vec{x}|$, and $\vec{A}(\vec{x})$ (potential vector) is a vector of functions of position operators.

(1) Here we define the operator

$$\vec{v} = \frac{1}{m}(\vec{p} - \vec{A}(\vec{x})), \tag{2}$$

for which we can determine the commutation relations between each of its components and the components of the position operator as follows:

$$[v^i, x^j] = \frac{1}{m}[p^i - A^i(\vec{x}), x^j] = \frac{1}{m}[p^i, x^j] = -\frac{i\hbar}{m}\delta^{ij}. \tag{3}$$

(2) The commutation between any two components of the operator v^i is given by

$$\begin{aligned} \left[v^i, v^j\right] &= \left[\frac{1}{m}\left(p^i - A^i(\vec{x})\right), \frac{1}{m}\left(p^j - A^j(\vec{x})\right)\right], \\ &= -\frac{1}{m^2}\left(\left[p^i, A^j(\vec{x})\right] + \left[A^j(\vec{x}), p^i\right]\right), \\ &= \frac{i\hbar}{m^2}\left(\partial^i A^j(\vec{x}) - \partial^j A^i(\vec{x})\right). \end{aligned} \tag{4}$$

(3) The Heisenberg equation of motion for the position operator $\vec{x}$ is

$$\frac{\mathrm{d}x^j}{\mathrm{d}t} = \frac{i}{\hbar}\left[H, x^j\right] = \frac{i}{2m\hbar}\left[p^i p_i - p^i A_i(\vec{x}) - A^i(\vec{x})p_i - A^i(\vec{x})A_i(\vec{x}) - \frac{e^2}{r}, x^j\right], \tag{5}$$

$$= \frac{i}{2m\hbar}\left(p_i[p^i, x^j] + [p_i, x^j]p^i - A_i(\vec{x})[p^i, x^j] - [p_i, x^j]A^i(\vec{x})\right), \tag{6}$$

$$= \frac{1}{m}(p^j - A^j(\vec{x})) = v^j. \tag{7}$$

(4) Assuming that $\vec{A}(\vec{x})$ is given by

$$\vec{A}(\vec{x}) = \frac{B}{2}\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \tag{8}$$

we can write the Hamiltonian H as

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{B^2}{8m}\left(x_1^2 + x_2^2\right) - \frac{B}{2m}\left(x_1 p_2 - x_2 p_1\right) + \frac{p_3^2}{2m} - \frac{e^2}{r}. \tag{9}$$

Using the fact that the third component orbital angular momentum L_3 can be written as

$$L_3 = x_1 p_2 - x_2 p_1, \tag{10}$$

we can write the Hamiltonian H in terms of the requested operators:

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{B^2}{8m}\left(x_1^2 + x_2^2\right) - \frac{B}{2m}L_3 + \frac{p_3^2}{2m} - \frac{e^2}{r}. \tag{11}$$

(5) In the case where $e = 0$, and Eq. (9) still holds, the Hamiltonian H can be split into a part that depends only on the coordinates x_1 and x_2 and a part which depends only on the coordinate x_3 (and the respective momenta) as

$$H = H_{12}(x_1, x_2, p_1, p_2) + H_3(x_3, p_3), \tag{12}$$

$$H_{12}(x_1, x_2, p_1, p_2) = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{B^2}{8m}\left(x_1^2 + x_2^2\right) - \frac{B}{2m}L_3, \tag{13}$$

$$H_3(x_3, p_3) = \frac{p_3^2}{2m}. \tag{14}$$

It can now be seen that H_3 is a free-particle Hamiltonian which we know has a continuous energy

$$E_{k_3}^3 = \frac{\hbar^2 k_3^2}{2m}, \tag{15}$$

where $\pm\hbar k_3$ are the eigenvalues of the momentum operator p_3.

Let us now write H_{12} in the form

$$H_{12}(x_1, x_2, p_1, p_2) = H_{\text{osc.}} + \frac{1}{2}m\omega^2 x_2^2 - \omega L_3, \tag{16}$$

where

$$\omega = \frac{B}{2m}, \tag{17}$$

and

$$H_{\text{osc}} = \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + \frac{p_2^2}{2m}, \tag{18}$$

is the Hamiltonians of a harmonic oscillators defined in the dimensions x_1 and x_2. For such a Hamiltonian we know that the eigenvalue spectrum is

$$E_n^{\text{osc}} = \hbar\omega\left(n_1 + \frac{1}{2}\right) + \hbar\omega\left(n_2 + \frac{1}{2}\right) = \hbar\omega(n+1), \tag{19}$$

where $n_1 + n_2 = n$.

Next, we note that H_{osc} is invariant under rotation:

$$[H_{\text{osc}}, L_3] = 0, \tag{20}$$

which means that we the eigenstates of H_{12} can be chosen as simultaneous eigenstates of H and L_3. The eigenvalues are then ths sum of the eigenstates of a two-dimensional harmonic oscillator with quantum number n, and the eigenstates of the third component of the angular momentum with quantum number l_3.

Summing the eigenvalues discussed above, we can write the spectrum of eigenvalues of H as

$$E_{n,l_3,k_3} = \hbar\omega(n + 1 - l_3) + \frac{\hbar^2 k_3^2}{2m}, \tag{21}$$

with integer n and l_3 and continuous k_3.

(6) See Sect. 9.2.2 of the textbook.

(7) The spectrum of the Hamiltonian in the x_1, x_2 plane is given by

$$E_{n,l_3} = \hbar\omega(n + 1 - l_3). \tag{22}$$

This is infinitely degenerate because the spectrum of n includes all the positive integers and the spectrum of l_3 all the integers, so given any eigenstate with eigenvalue E_{n,l_3} we can obtain an infinite number of eigenstates with the same eigenvalue by simply picking any other eigenstate in which the values of n and l_3 are increased by the same amount: Eq. (23) implies that $E_{n,l_3} = E_{n+k,l_3+k}$. The degeneracy of the spectrum given in Eq. (22) is of course the same.

(8) We now consider the case in which $e \neq 0$ and $\vec{A}(\vec{x})$ is given by Eq. (9). Treating $\vec{A}(\vec{x})$ as a first order perturbation in B, we can write the Hamiltonian as

$$H = H_H - \frac{B}{2m} L_3 + O(B^2), \tag{23}$$

where H_H is the Hamiltonian of the Hydrogen atom

$$H_H = \frac{\vec{p}^2}{2m} - \frac{e^2}{r}, \tag{24}$$

and we have neglected terms of order B^2 and higher.

The spectrum of the hydrogen atom is

$$E_n = -\frac{me^4}{2\hbar^2 n^2}. \tag{25}$$

The hydrogen eigenstates can be chosen as angular momentum eigenstates $|nll_3\rangle$, with $l \leq n-1$, $-l \leq l_3 \leq l$. In such a state the first-order perturbation is given by

$$\Delta E_{nll_3} = -\frac{B}{2m} \langle nll_3 | L_3 | nll_3 \rangle == -\frac{B}{2m} \hbar l_3. \tag{26}$$

(9) Let us consider an eigenfunction $\psi(\vec{x})$ of the Hamiltonian H. The Schrödinger equation for this eigenfunction can be written as

$$H\psi(\vec{x}) = E\psi(\vec{x}). \tag{27}$$

Transforming both the Hamiltonian and the eigenstate as suggested in the hint gives

$$e^{-i\phi(\vec{x})} H e^{i\phi(\vec{x})} e^{-i\phi(\vec{x})} \psi(\vec{x}) = E e^{-i\phi(\vec{x})} \psi(\vec{x}), \tag{28}$$

where we now see that $e^{-i\phi(\vec{x})}\psi(\vec{x})$ is an eigenfunction of the Hamiltonian

$$e^{-i\phi(\vec{x})} H e^{i\phi(\vec{x})} = e^{-i\phi(\vec{x})} \frac{(\vec{p} - \vec{A}(\vec{x}))^2}{2m} e^{i\phi(\vec{x})} - \frac{e^2}{r}. \tag{29}$$

Here we have

$$e^{-i\phi(\vec{x})} (\vec{p} - \vec{A}(\vec{x})) e^{i\phi(\vec{x})} = e^{-i\phi(\vec{x})} (-i\hbar\vec{\nabla} - \vec{A}(\vec{x})) e^{i\phi(\vec{x})}, \tag{30}$$

$$= -i\hbar\vec{\nabla} - \vec{A}(\vec{x}) + \hbar\vec{\nabla}\phi(\vec{x}), \tag{31}$$

$$= \vec{p} - \left(\vec{A}(\vec{x}) + \vec{\nabla}\Lambda(\vec{x})\right), \tag{32}$$

where in the last step we used $\Lambda(\vec{x}) \equiv -\hbar\phi(\vec{x})$.

This shows that two Hamiltonians of the form Eq. (1), having two different potential vectors $A_1(\vec{x})$ and $A_2(\vec{x})$ such that

$$A_1(\vec{x}) = A_2(\vec{x}) + \vec{\nabla}\Lambda(\vec{x}) \tag{33}$$

are unitarily equivalent.

(10) In the previous exercise we have shown unitary equivalence between Hamiltonians of the form Eq. (1) and where the potential vectors $A(\vec{x})$ are related as in Eq. (34). Here we will use this to determine the spectrum of Eq. (1) if $A(\vec{x})$ is

$$\vec{A}(\vec{x}) = \frac{B}{2} \begin{pmatrix} x_2 \\ x_1 \\ 0 \end{pmatrix}. \tag{34}$$

Namely, choosing $\Lambda(\vec{x}) = -\frac{B}{2}x_1 x_2$ we see that

$$\vec{A}'(\vec{x}) = \vec{A}(\vec{x}) - \frac{B}{2}\vec{\nabla}(x_1 x_2) = \vec{0}, \tag{35}$$

meaning the spectrum of the Hamiltonian Eq. (1) with $A(\vec{x})$ Eq. (35) is the same as that of the Hamiltonian H_H Eq. (25), which we recognise as the Hamiltonian of the hydrogen atom of which the spectrum is given by Eq. (26).

(11) The spectrum of Eq. (27) depends on both l_3 and n. Here we have, for each combination of n and l_3, possible integer values of the azimuthal quantum number l in the range $|l_3| \leq l \leq n-1$, and thus the degeneracy is

$$d = n - |l_3|. \tag{36}$$