QUANTUM MECHANICS II EXAM

16 September 2021

Answers sheet

Consider a three-dimensional particle whose dynamics is described by the Hamiltonian

$$H = \frac{\vec{p}^2}{2m} + \frac{1}{2}m\left[\frac{\omega_1^2 + \omega_2^2}{2}(x_1)^2 + (\omega_1^2 - \omega_2^2)x_1 x_2 + \frac{\omega_1^2 + \omega_2^2}{2}(x_2)^2 + \omega_3^2(x_3)^2\right] \quad (1)$$

where x_1, x_2, and x_3 are the three components of the position operator for the particle and $\vec{p}$ is the corresponding vector of momentum operators.

(1) Let us consider a system described by Eq. (1), assuming $\omega_1 = \omega_2 = \omega_3 = \omega$ we find

$$H = \frac{\vec{p}^2}{2m} + \frac{1}{2}m\omega^2\vec{x}^2 = \sum_{i=1}^{3}\left(\frac{(p_i)^2}{2m} + \frac{1}{2}m\omega^2(x_i)^2\right) \quad (2)$$

which we recognise as the Hamiltonian of a harmonic oscillator in three dimensions, or, alternatively, three independent one dimensional oscillators. The energy eigenvalue spectrum of a one-dimensional harmonic oscillator is a well-known result:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right). \quad (3)$$

From this it follows that the eigenvalue spectrum of the Hamiltonian Eq. (2) is

$$E_{n_1,n_2,n_3} = \hbar\omega\left(n_1 + n_2 + n_3 + \frac{3}{2}\right) = \hbar\omega\left(N + \frac{3}{2}\right), \quad (4)$$

with $N = n_1 + n_2 + n_3$.

(2) For a given N we have $n_1 \in \{0, 1, \dots, N\}$, after choosing a value for n_1 we have $n_2 \in \{0, 1, \dots, N - n_1\}$, and after choosing a value for n_2, $n_3 = N - n_1 - n_2$ is fixed. Therefore, the degeneracy d of the Hamiltonian Eq. (2) is

$$d = \sum_{n_1=0}^{N}(N - n_1 + 1) = \sum_{n_1=0}^{N}(N+1) - \sum_{n_1=0}^{N}n_1 = (N+1)(N+1) - \frac{1}{2}N(N+1) = \frac{1}{2}(N+1)(N+2). \quad (5)$$

(3) The ground state of the one-dimensional harmonic oscillator is

$$u_0(x) = N_0\exp\left(-\frac{x^2 m\omega}{2\hbar}\right), \quad (6)$$

with $N_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}$. Because the Hamiltonian of our three-dimensional harmonic oscillator is equivalent to that of three independent one-dimensional oscillators, we can write the ground state of the Hamiltonian Eq. (2) as

$$\psi_0(\vec{x}) = u_0(x_1)u_0(x_2)u_0(x_3) = N_0\exp\left(-\frac{|\vec{x}|^2 m\omega}{2\hbar}\right). \quad (7)$$

(4) Now let us again assume that $\omega_1 = \omega_2 = \omega$, but this time $\omega_3 \neq \omega$. In this case the Hamiltonian becomes

$$H = \frac{\vec{p}^2}{2m} + \frac{1}{2}m\omega^2\left((x_1)^2 + (x_2)^2\right) + \frac{1}{2}m\omega_3^2(x_3)^2, \tag{8}$$

with the corresponding energy eigenvalue being

$$E_{n_1,n_2,n_3} = E_{n_1,n_2} + E_{n_3} = \hbar\omega\left(n_1 + n_2 + 1\right) + \hbar\omega_3\left(n_3 + \frac{1}{2}\right), \tag{9}$$

where $E_{n_1,n_2} = \hbar\omega\left(n_1 + n_2 + 1\right)$ and $E_{n_3} = \hbar\omega_3\left(n_3 + \frac{1}{2}\right)$. From this it immediately follows that the degeneracy of E_{n_1,n_2} is $d = n_1 + n_2 + 1$, while E_{n_3} is non-degenerate. Assuming furthermore that ω and ω_3 are incommensurable, we find the degeneracy of the eigenvalue spectrum of the Hamiltonian Eq. (8) to be

$$d = N + 1, \tag{10}$$

where

$$N = n_1 + n_2. \tag{11}$$

(5) Let us now consider the Hamiltonian $H' = T_3^{-1}(\delta)HT_3(\delta)$, with the Hamiltonian H as in Eq. (8) and where $T_3(\delta)$ is the operator that realizes a translation of length δ along the x_3 axis. For the Hamiltonian H' we find

$$H' = T_3^{-1}(\delta)HT_3(\delta), \tag{12}$$

$$= \frac{\vec{p}^2}{2m} + \frac{1}{2}m\omega^2\left((x_1)^2 + (x_2)^2\right) + \frac{1}{2}m\omega_3^2(x_3 - \delta)^2. \tag{13}$$

This has the same form of Eq. (8), except that the oscillator with frequency ω_3 is now centered about δ rather than in the origin. But of course the spectrum of an oscillator does not depend on the point in space where it is centered, so the spectrum is the same as given in Eq. (9), and the degeneracy is also the same.

The same result is found simply observing that the operators H and H' are unitarily equivalent, and unitarily equivalent operators have the same spectrum of eigenvalues.

The wave function of the fundamental state of H' is

$$\psi'(\vec{x}) = \langle x|T_3^{-1}(\delta)|\psi\rangle = u_0(x_1)u_0(x_2)u_0(x_3 - \delta). \tag{14}$$

(6) See section 11.2.4 of the textbook.

(7) In the case where $\omega_1 \neq \omega_2$, the Hamiltonian of Eq. (1) can be written as the sum of three commuting Hamiltonians:

$$H = \frac{\vec{p}^2}{2m} + \frac{1}{2}m\left[\frac{1}{2}\omega_1^2\left((x_1)^2 + (x_2)^2 + 2x_1x_2\right) + \frac{1}{2}\omega_2^2\left((x_1)^2 + (x_2)^2 - 2x_1x_2\right) + \omega_3^2(x_3)^2\right], \tag{15}$$

$$= \frac{\vec{p}^2}{2m} + \frac{1}{2}m\left[\omega_1^2(\bar{x}_1)^2 + \omega_2^2(\bar{x}_2)^2 + \omega_3^2(x_3)^2\right], \tag{16}$$

where $\bar{x}_1 = (x_1 + x_2)/\sqrt{2}$, and $\bar{x}_2 = (x_1 - x_2)/\sqrt{2}$.

From this it follows that the spectrum of the Hamiltonian is simply

$$E_{n_1,n_2,n_3} = \hbar\omega_1\left(n_1 + \frac{1}{2}\right) + \hbar\omega_2\left(n_2 + \frac{1}{2}\right) + \hbar\omega_3\left(n_3 + \frac{1}{2}\right). \tag{17}$$

(8) Still in the case in which $\omega_1 \neq \omega_2$, we consider the term proportional to $x_1 x_2$ as a perturbation. Let us furthermore assume that $\omega_3 \gg \omega_1$ and $\omega_3 \gg \omega_2$. Treating the term proportional to $x_1 x_2$ perturbatively relies on the assumption that $|\omega_1^2 - \omega_2^2| \ll \omega_1^2 + \omega_2^2$.

Let us define

$$\Omega_\pm = \sqrt{(\omega_1^2 \pm \omega_2^2)/2}, \tag{18}$$

such that the Hamiltonian can be written as

$$H = H_0 + V, \tag{19}$$

with

$$H_0 = \frac{\vec{p}^2}{2m} + \frac{1}{2}m \left[\Omega_+^2 (x_1)^2 + \Omega_+^2 (x_2)^2 + \omega_3^2 (x_3)^2\right], \tag{20}$$

and the term proportional to $x_1 x_2$:

$$V = \frac{1}{2}m(\omega_1^2 - \omega_2^2)x_1 x_2 = \frac{\hbar\Omega_-^2}{2\Omega_+}\left(a_1 + a_1^\dagger\right)\left(a_2 + a_2^\dagger\right). \tag{21}$$

The unperturbed Hamiltonian then concides with that of question (4), with $\omega = \Omega_+$.

It is clear that the first-order correction for the ground state is zero and thus that for the ground state we have

$$E^{(0)} = \hbar\Omega_+ + \frac{1}{2}\hbar\omega_3. \tag{22}$$

The assumption that $\omega_3 \gg \omega_1, \omega_2$ implies that the first excited state is that in which the oscillator with frequency ω_3 remains in the ground state, while $N = 1$, with N defined in Eq. (11). This first excited state is therefore doubly degenerate and we have that

$$\langle N = 1|V|N = 1\rangle, \tag{23}$$

$$= \left\langle N = 1 \left| \frac{\hbar\Omega_-^2}{2\Omega_+}(a_1 + a_1^\dagger)(a_2 + a_2^\dagger) \right| N = 1 \right\rangle, \tag{24}$$

$$= \frac{\hbar\Omega_-^2}{2\Omega_+}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{25}$$

where by $|N = 1\rangle$ we have denoted a two-component state vector in the subspace of degenerate states, which we have evaluated in the basis

$$|N = 1\rangle = \begin{pmatrix} |n_1 = 1, n_2 = 0\rangle \\ |n_1 = 0, n_2 = 1\rangle \end{pmatrix}. \tag{26}$$

The eigenvalues of Eq. (25) can easily be obtained, from which it then follows that for the first excited state we have

$$E^{(1)} = 2\hbar\Omega_+ + \frac{1}{2}\hbar\omega_3 + \Delta E = 2\hbar\Omega_+ + \frac{1}{2}\hbar\omega_3 \pm \frac{\hbar\Omega_-^2}{2\Omega_+}. \tag{27}$$

(9) The unitary transformation that realizes the change of coordinates in exercise (7) can be understood as a rotation by an angle $\pi/4$ in the (x_1, x_2) plane. Once this is appreciated, the Hermitian operator that generates this transformation can be written as

$$R_{\pi/4} = \exp\left(-\frac{i}{\hbar}\frac{\pi}{4}(x_1 p_2 - x_2 p_1)\right) \tag{28}$$

$$= \exp\left(-\frac{i}{\hbar}\frac{\pi}{4}L_z\right) \tag{29}$$

$$= \exp\left(\frac{\pi}{4}\frac{\partial}{\partial\phi}\right), \tag{30}$$

where ϕ is the polar angle in the (x_1, x_2) plane.

(10) The exact result for the energy spectrum is given in Eq. (17). From this expression we find that the energy eigenvalue of the ground state is

$$E_{0,0,0} = \frac{1}{2}\hbar \left(\omega_1 + \omega_2 + \omega_3 \right), \tag{31}$$

$$= \frac{1}{2}\hbar\omega_3 + \frac{1}{2}\hbar \left(\sqrt{\Omega_+^2 + \Omega_-^2} + \sqrt{\Omega_+^2 - \Omega_-^2} \right), \tag{32}$$

$$= \frac{1}{2}\hbar\omega_3 + \hbar\Omega_+ + \mathcal{O}\left(\frac{\Omega_-^2}{\Omega_+} \right). \tag{33}$$

While that of the first excited state is

$$E_{0,1,0} = \frac{1}{2}\hbar\omega_1 + \frac{3}{2}\hbar\omega_2 + \frac{1}{2}\hbar\omega_3, \tag{34}$$

$$= \frac{1}{2}\hbar\sqrt{\Omega_+^2 + \Omega_-^2} + \frac{3}{2}\hbar\sqrt{\Omega_+^2 - \Omega_-^2} + \frac{1}{2}\hbar\omega_3, \tag{35}$$

$$= 2\hbar\Omega_+ - \frac{\hbar\Omega_-^2}{2\Omega_+} + \frac{1}{2}\hbar\omega_3 + \mathcal{O}\left(\frac{\Omega_-^2}{\Omega_+} \right), \tag{36}$$

and

$$E_{1,0,0} \approx 2\hbar\Omega_+ + \frac{\hbar\Omega_-^2}{2\Omega_+} + \frac{1}{2}\hbar\omega_3 + \mathcal{O}\left(\frac{\Omega_-^2}{\Omega_+} \right), \tag{37}$$

thus showing equivalence between the results of exercises (7) and (8) for a first order perturbation where the perturbation parameter is proportional to $\frac{\Omega_-^2}{\Omega_+}$.

(11) The eigenvalue spectrum is the same as for a single three-dimensional particle and is given by Eq. (4), but with the extra condition $N \geq 3$ because the ground state is the lowest state such that $n_1 \neq n_2 \neq n_3$.

For a system of three fermions in a single dimension, and with the fermions in the same spin state, the spatial part of the wave function must be fully antisymmetric. As a result of this, the ground state energy eigenvalue is $E_{0,1,2}$ and also non-degenerate, the energy eigenvalue of the first excited state is $E_{0,1,3}$ and non-degenerate, and finally, the energy eigenvalue of the second excited state is $E_{0,2,3} = E_{0,1,4}$ and doubly degenerate. Indeed, note that antisimmetrization removes exchange degeneracy, so energy levels E_{ijk} that differ by a permutation of the indices ijk are no longer degenerate.