QUANTUM FIELD THEORY I

Solution

January 16, 2020

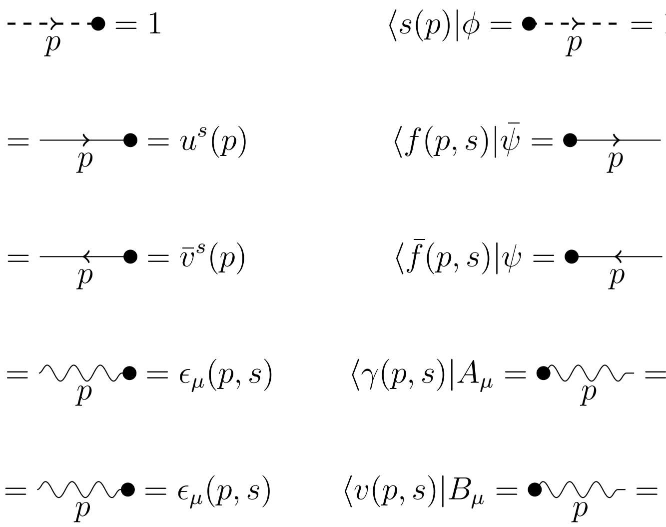
Consider a theory given by the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_s^2\phi^2\right) + \bar{\psi}(i\partial\!\!\!/ - m_f)\psi + \frac{g}{4}\phi F_{\mu\nu}F^{\mu\nu} - g'\bar{\psi}\gamma^\mu\psi B_\mu, \quad (1)$$

where $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ is the Maxwell field tensor, ϕ is a real scalar field, ψ is a Dirac fermion field and B_μ is an external vector field.

(1) Let us first write down the Feynman rules for this theory.

(a) The Feynman rules for the external lines can be derived by acting the fields on the initial- and final-state particles.



$\phi|s(p)\rangle = 1$ $\qquad$ $\langle s(p)|\phi = 1$

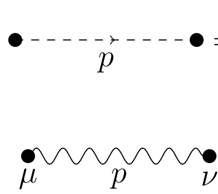
$\psi|f(p,s)\rangle = u^s(p)$ $\qquad$ $\langle f(p,s)|\bar{\psi} = \bar{u}^s(p)$

$\bar{\psi}|\bar{f}(p,s)\rangle = \bar{v}^s(p)$ $\qquad$ $\langle \bar{f}(p,s)|\psi = v^s(p)$

$A_\mu|\gamma(p,s)\rangle = \epsilon_\mu(p,s)$ $\qquad$ $\langle \gamma(p,s)|A_\mu = \epsilon^*_\mu(p,s)$

$B_\mu|v(p,s)\rangle = \epsilon_\mu(p,s)$ $\qquad$ $\langle v(p,s)|B_\mu = \epsilon^*_\mu(p,s).$

(b) The propagators can be computed by contracting the fields.

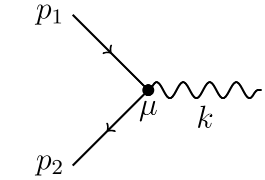


$$\text{Scalar}: \quad = \frac{i}{p^2 - m^2 + i\epsilon} \quad (2)$$

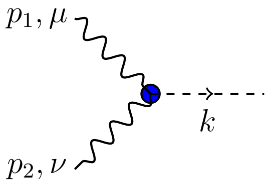
$$\text{Photon}: \quad = -\frac{i}{p^2 + i\epsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right]. \quad (3)$$

In the expression of the photon propagator, ξ parametrizes a set of covariant gauges. The Feynman gauge $\xi = 1$ is considered as a correct solution.

(c) The interaction terms in the Lagrangian introduces the following vertices:



$$= -ig'\gamma^{\mu} \tag{4}$$



$$= -ig\left((p_1 p_2)g^{\mu\nu} - p_1^{\mu} p_2^{\nu}\right) \tag{5}$$

The coupling g' is dimensionless, but the coupling g has a dimension of $[m]^{-1}$ and the theory is thus nonrenormalizable

(2) The energy-momentum tensor of the theory can be written as:

$$T^{\mu}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A_{\lambda})}\partial_{\nu} A_{\lambda} + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\partial_{\nu}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\psi)}\partial_{\nu}\psi - \delta^{\mu}_{\nu}\mathcal{L} \tag{6}$$

$$= -(1 - g\phi)F^{\mu\lambda}\partial_{\nu} A_{\lambda} + \partial^{\mu}\phi\partial_{\nu}\phi - i\bar{\psi}\gamma^{\mu}\partial_{\nu}\psi - \delta^{\mu}_{\nu}\mathcal{L}. \tag{7}$$

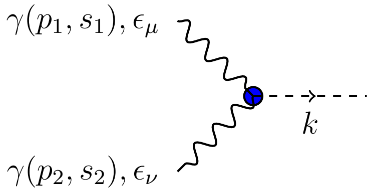
From this, we can now derive the Hamiltonian density which is defined as $\mathcal{H} = T^{00}$,

$$\mathcal{H} = -(1 - g\phi)F^{0\lambda}\dot{A}_{\lambda} + \dot{\phi}^2 - i\psi^{\dagger}\dot{\psi} - g^{00}\mathcal{L} \tag{8}$$

$$= \frac{1}{2}\left(\dot{\phi}^2 + \vec{\nabla}\phi \cdot \vec{\nabla}\phi + m_s^2\phi^2\right) + i\bar{\psi}\vec{\gamma} \cdot \vec{\nabla}\psi + m_f\bar{\psi}\psi + (1 - g\phi)\left(F^{\mu\nu}F_{\mu\nu} - F^{0\mu}\dot{A}_{\mu}\right). \tag{9}$$

A simpler form of the electromagnetic contribution to the energy-momentum tensor can be obtained through the Belinfante construction, which then leads to the covaraint result $T^{\mu\nu} = F^{\mu\alpha}F_{\nu\alpha} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, leading to the Hamiltonian $\mathcal{H} = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)$.

(3) Let us now compute the unpolarized squared amplitude for the process $\gamma(p_1) + \gamma(p_2) \to \phi(k)$. The lowest non-trivial diagram is given by:



$$= i\mathcal{M}_1(\gamma\gamma \to \phi). \tag{10}$$

Using the Feynman rules derived in the previous section, we have:

$$i\mathcal{M}_1(\gamma\gamma \to \phi) = -(ig)\epsilon_{\mu}(p_1)\epsilon_{\nu}(p_2)\mathrm{M}^{\mu\nu}, \tag{11}$$

where $\mathrm{M}^{\mu\nu} = (p_1 p_2) g^{\mu\nu} - p_1^\mu p_2^\nu$. Hence, squaring the above amplitude yields,

$$|\mathcal{M}_1(\gamma\gamma \to \phi)|^2 = g^2 \epsilon_\mu(p_1)\epsilon_\nu(p_2)\epsilon^*_\rho(p_1)\epsilon^*_\beta(p_2)\mathrm{M}^{\mu\nu}\mathrm{M}^{\rho\beta}. \tag{12}$$

Averaging over the initial states and summing over the polarizations using the identity, we have

$$|\bar{\mathcal{M}}_1(\gamma\gamma \to \phi)|^2 = \frac{1}{4}\sum_{\text{pol.}} |\mathcal{M}_1(\gamma\gamma \to \phi)|^2 \tag{13}$$

$$= \frac{g^2}{4} g_{\mu\rho} g_{\nu\beta} \mathrm{M}^{\mu\nu}\mathrm{M}^{\rho\beta} \tag{14}$$

$$= \frac{g^2}{4} \mathrm{M}^{\mu\nu}\mathrm{M}_{\mu\nu} \tag{15}$$

$$= \frac{g^2}{2} (p_1 \cdot p_2)^2. \tag{16}$$

In terms of the Mandelstam variables, the unpolarized squared amplitude is given by the following

$$|\bar{\mathcal{M}}_1(\gamma\gamma \to \phi)|^2 = \frac{g^2}{8} s^2. \tag{17}$$

(4) In order to compute the cross-section of the production of a scalar field from an annihilation of two photons, let us first derive the 1-body phase space given by the following expression:

$$\mathrm{d}\Phi(\gamma\gamma \to \phi) = \frac{\mathrm{d}^3 k}{(2\pi)^3 2E_k} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k). \tag{18}$$

Integrating over k in the center of mass of the two colliding photons yields

$$\mathrm{d}\Phi(\gamma\gamma \to \phi) = \frac{\pi}{m_s} \delta(\sqrt{s} - m_s) \tag{19}$$

$$= 2\pi \delta(s - m_s^2). \tag{20}$$

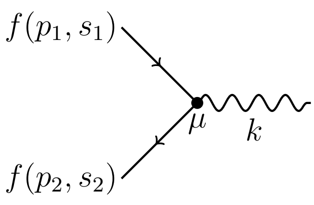
On the other hand, the flux is given by:

$$\mathcal{F}(\gamma\gamma \to \phi) = 4(p_1 \cdot p_2) = 2s. \tag{21}$$

Combining all these results, one can finally derive the final expression of the cross section given as

$$\sigma(\gamma\gamma \to \phi) = \frac{g^2}{8} m_s^2 \pi \delta(s - m_s^2). \tag{22}$$

(5) The lowest perturbative order of the process $f(p_1) + \bar{f}(p_2) \to B(k)$ is given by the following diagram:



$$= i\mathcal{M}_2(f\bar{f} \to B), \tag{23}$$

where the amplitude is expressed as

$$i\mathcal{M}_2(f\bar{f} \to B) = (-ig')\epsilon^*_\mu(p_1)u(p_1)\gamma^\mu\bar{v}(p_2). \tag{24}$$

Hence, taking the square of the above amplitude yields,

$$|\mathcal{M}_2(f\bar{f} \to B)|^2 = (g')^2\epsilon^*_\mu(p_1)\epsilon_\nu(p_1)\left(u(p_1)\gamma^\mu\bar{v}(p_2)v(p_2)\gamma^\nu\bar{u}(p_1)\right). \tag{25}$$

Averaging over the initial state and summing over the polarization of the final states leads to the following results,

$$|\bar{\mathcal{M}}_2(f\bar{f} \to B)|^2 = -\frac{(g')^2}{4}u(p_1)\gamma^\mu\bar{v}(p_2)v(p_2)\gamma_\mu\bar{u}(p_1) \tag{26}$$

$$= -\frac{(g')^2}{4}\mathrm{Tr}\left((\not{p}_1 + m_f)\gamma^\mu(\not{p}_2 - m_f)\gamma_\mu\right) \tag{27}$$

$$= \frac{(g')^2}{2}\left(\mathrm{Tr}(\not{p}_1\not{p}_2) + 8m_f^2\right) \tag{28}$$

$$= 2(g')^2\left((p_1p_2) + 2m_f^2\right), \tag{29}$$

where in the second line, we have used the fact that $\gamma^\mu\gamma_\mu = 4$ and $\gamma^\mu\not{p}_2\gamma_\mu = -2\not{p}_2$. Expressed in terms of the Mandelstam variables, we get

$$|\bar{\mathcal{M}}_2(f\bar{f} \to B)|^2 = (g')^2\left(s + 2m_f^2\right). \tag{30}$$

The Lorentz invariant phase space is the same as in question(4)

$$\mathrm{d}\Phi(f\bar{f} \to B) = 2\pi\delta(s - m_B^2) \tag{31}$$

while the flux factor is now

$$\mathcal{F}(f\bar{f} \to B) = 4E^2 2\frac{\sqrt{E^2 - m_f^2}}{2E} = 2s\sqrt{1 - \frac{4m^2}{s}}, \tag{32}$$

where $E = \sqrt{s}/2$ is the energy of the incoming (massive) fermions. The total cross section is therefore given by,

$$\sigma = (g')^2\left(1 + \frac{2m_f^2}{s}\right)\frac{1}{\sqrt{1 - \frac{4m^2}{s}}}\pi\delta(s - m_B^2). \tag{33}$$

(6) Both processes are $2 \to 1$. Hence the cross-section can depend on 9 momentum components, minus six Lorenz conditions, giving three scalar products: $p_1 \cdot p_2$, $p_1 \cdot k$, $p_2 \cdot p_2$. Momentum conservation then expresses all these scalar productes in terms of a single one (which can be taken to be the Mandelstam invariant s). However, because there are four momentum conservation conditions and only three scalar products, the system is overconstrained, and the cross-section does not depend on any kinematic variable, rather, it constrains the remaining scalar product in terms of a mass through a delta function. Hence, the fact that the cross-section is proportional to a delta of $s - m^2$ could be established without performing any calculation. Furthermore, the fact that the cross-section Eq. (22) is proportional to $g^2m_s^2$ follows from dimensional analysis, recalling that the constant g has dimensions of mass^{-1}, and so does the fact that the cross-section Eq. (33) is proportional to the dimensionless g^2, up to corrections of order $\frac{m_f^2}{s}$.

(7) Assuming that each of the two photons in (5) is part of a flux of incoming fermions with momentum $p_i = x_i p_{\max}$, and defining

$$\sigma_0 = (g')^2 \left(1 + \frac{2m_f^2}{s}\right) \frac{1}{\sqrt{1 - \frac{4m^2}{s}}} \pi, \tag{34}$$

and

$$\tau = \frac{(m_B^2 - 2m_f^2)}{p_1^{\max} \cdot p_2^{\max}} \tag{35}$$

the cross section is

$$\begin{aligned} \sigma &= \sigma_0 \int_0^1 dx_1 \int_0^1 dx_2 p(x_1) p(x_2) \delta\left(x_1 x_2 p_1^{\max} \cdot p_2^{\max} - (m_B^2 - 2m^2 f)\right) \\ &= \sigma_0 \int_0^1 dx_1 \int_0^1 dx_2 p(x_1) p(x_2) \delta(x_1 x_2 - \tau) \\ &= \sigma_0 \int_\tau^1 \frac{dx_2}{x_2} p\left(\frac{\tau}{x_2}\right) p(x_2). \end{aligned} \tag{36}$$