Solution of the exam of Theoretical Physics of January 23 2024

Real scalar field:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ipx} + a_p^\dagger e^{ipx} \right) . \tag{1}$$

Spinor field:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1}^{2} \left(a_p^s u^s(p) e^{-ipx} + b_p^{s\dagger} v^s(p) e^{ipx} \right) . \tag{2}$$

1. The energy-momentum tensor is defined as

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial_\nu \psi - \delta^\mu{}_\nu \mathcal{L} \, . \tag{3}$$

Using

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar{\psi}\gamma^\mu \, , \tag{4}$$

we find that

$$T^\mu{}_\nu = \partial^\mu \phi \partial_\nu \phi + i\bar{\psi}\gamma^\mu \partial_\nu \psi - \delta^\mu{}_\nu \mathcal{L} \, . \tag{5}$$

The Hamiltonian density is

$$\begin{aligned}
\mathcal{H} = T^{00} &= \partial^0\phi\partial^0\phi + i\psi^\dagger\partial^0\psi - \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m\phi^2\right) - \bar{\psi}\left(i\partial\!\!\!/ + ig(a+b\gamma_5)\phi\right)\psi \\
&= \dot{\phi}^2 + i\psi^\dagger\dot{\psi} - \frac{1}{2}\left(\dot{\phi}^2 + \partial_i\phi\partial^i\phi - m\phi^2\right) - \bar{\psi}\left(i\gamma^0\partial_0 + i\gamma^i\partial_i + ig(a+b\gamma_5)\phi\right)\psi \\
&= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi + \frac{m}{2}\phi^2 - i\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi - ig(a+b\gamma_5)\phi\bar{\psi}\psi
\end{aligned} \tag{6}$$

2. The only internal symmetry of the theory is

$$\psi \to \psi' = e^{-i\theta}\psi \, , \tag{7}$$
$$\bar{\psi} \to \bar{\psi}' = e^{i\theta}\bar{\psi} \, . \tag{8}$$

The associated Noether current is

$$\mathcal{J}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta\psi = \bar{\psi}\gamma^\mu\psi \, , \tag{9}$$

where we used that

$$\Delta\psi = \frac{\delta\psi}{\theta} = \frac{-i\theta\psi}{\theta} = -i\psi \, . \tag{10}$$

The classical conserved charge is

$$Q_{U(1)} = \int d^3x \, \mathcal{J}^0(x) = \int d^3x \, \bar{\psi}(x)\gamma^0\psi(x) = \int d^3x \, \psi^\dagger(x)\psi(x) \, . \tag{11}$$

In order to write thecharge in terms of creation and annihilation operators we insert Eq. (2) in Eq. (11), finding

$$Q_{U(1)} = \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^3\sqrt{2E_p 2E_{p'}}} \sum_{ss'} \left(a_p^{s\dagger} u^{s\dagger}(p) e^{ipx} + b_p^s v^{s\dagger}(p) e^{-ipx} \right) \left(a_{p'}^{s'} u^{s'}(p') e^{-ip'x} + b_{p'}^{s'\dagger} v^{s'}(p') e^{ip'x} \right) \tag{12}$$

$$\begin{aligned}
= \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^3\sqrt{2E_p 2E_{p'}}} \sum_{ss'} \Big(& a_p^{s\dagger} a_{p'}^{s'} u^{s\dagger}(p) u^{s'}(p') e^{i(p-p')x} + a_p^{s\dagger} b_{p'}^{s'\dagger} u^{s\dagger}(p) v^{s'}(p') e^{i(p+p')x} \\
& + b_p^s a_{p'}^{s'} v^{s\dagger}(p) u^{s'}(p') e^{-i(p+p')x} + b_p^s b_{p'}^{s'\dagger} v^{s\dagger}(p) v^{s'}(p') e^{-i(p-p')x} \Big) \, .
\end{aligned} \tag{13}$$

Integrating over d^3x we get a delta that remove one of the integrals over momentum and therefore we find

$$Q_{U(1)} = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_{ss'} \Big(a_p^{s\dagger} a_p^{s'} u^{s\dagger}(p) u^{s'}(p) + a_p^{s\dagger} b_{-p}^{s'\dagger} u^{s\dagger}(p) v^{s'}(-p) + b_p^s a_{-p}^{s'} v^{s\dagger}(p) u^{s'}(-p) + b_p^s b_p^{s'\dagger} v^{s\dagger}(p) v^{s'}(p) \Big) . \quad (14)$$

Using

$$u^{s\dagger}(p) u^{s'}(p) = 2E_p \delta^{ss'} , \quad (15)$$

$$v^{s\dagger}(p) v^{s'}(p) = 2E_p \delta^{ss'} , \quad (16)$$

$$u^{s\dagger}(p) v^{s'}(-p) = 0 , \quad (17)$$

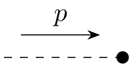
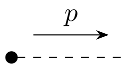
$$v^{s\dagger}(p) u^{s'}(-p) = 0 , \quad (18)$$

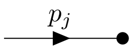
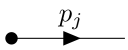
we find

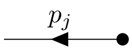
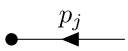
$$Q_{U(1)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1}^{2} \left(a_p^{s\dagger} a_p^s - b_p^{s\dagger} b_p^s \right) , \quad (19)$$

where we have anticommuted $b_p^{s\dagger}$ and b_p^s at the cost of removing an infinite constant.

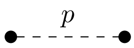
3.
- **External lines**

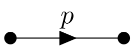
$\phi\,|s(p)\rangle =$  $= 1,$ $\qquad \langle s(p)|\,\phi =$  $= 1,$ (20)

$\psi\,|f(p_j,s)\rangle =$  $= u^s(p_j),$ $\qquad \langle f(p_j,s)|\,\overline{\psi} =$  $= \bar{u}^s(p_j),$ (21)

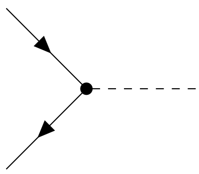
$\overline{\psi}\,|\bar{f}(p_j,s)\rangle =$  $= \bar{v}^s(p_j),$ $\qquad \langle f(p_j,s)|\,\overline{\psi} =$  $= v^s(p_j).$ (22)

- **Propagators**

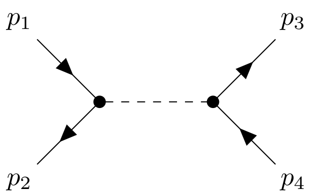
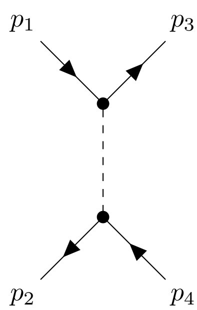
 $= \dfrac{i}{p^2 - m + i\epsilon},$ scalar propagator, (23)

 $= \dfrac{i(\not{p} + m)}{p^2 + i\epsilon},$ fermion propagator. (24)

- **Vertex**

 $= -g(a + b\gamma_5)\,.$ (25)

4. The diagrams that contribute to the process $f(p_1)\bar{f}(p_2) \to f(p_3)\bar{f}(p_4)$ at leading order are

 +  $= i\left(\mathcal{M}_s - i\mathcal{M}_t\right)\,.$ (26)

Note that with the Lagrangian written as in the assigment the mass parameter (with dimension [E]) is actually $\sqrt{m}$. (No penalty is given to those who overlooked this point). The relative sign between the two diagrams is a minus since to obtain the second one from the first one we have to exchange an antifermion from one bilinear with a fermion from the other bilinear.

Applying the Feynman rules it is easy to find

$$i\mathcal{M}^{(s)} = \bar{v}^{s_2}(p_2)\left(-g(a+b\gamma_5)\right) u^{s_1}(p_1) \frac{i}{(p_1+p_2)^2 - m} \bar{u}^{s_3}(p_3)\left(-g(a+b\gamma_5)\right) v^{s_4}(p_4) , \quad (27)$$

$$i\mathcal{M}^{(t)} = \bar{u}^{s_3}(p_3)\left(-g(a+b\gamma_5)\right) u^{s_4}(p_4) \frac{i}{(p_1-p_3)^2 - m} \bar{v}^{s_3}(p_3)\left(-g(a+b\gamma_5)\right) v^{s_4}(p_4) . \quad (28)$$

5. We have to compute the modulus squared of the unpolarized amplitude. Using

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p}\,, \quad \sum_s v^s(p)\bar{v}^s(p) = \not{p}\,, \tag{29}$$

since the fermion is massless, and averaging over initial polarizations, we obtain

$$\begin{aligned}
\overline{\sum}|\mathcal{M}|^2 &= \frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 = \\
&= \frac{g^2}{4}\Bigg[\frac{1}{((p_1+p_2)^2-m)^2}\mathrm{Tr}\Big(\not{p}_1(a+b\gamma_5)\not{p}_2(a+b\gamma_5)\Big)\mathrm{Tr}\Big(\not{p}_3(a+b\gamma_5)\not{p}_4(a+b\gamma_5)\Big) \\
&+ \frac{1}{((p_1-p_3)^2-m)^2}\mathrm{Tr}\Big(\not{p}_1(a+b\gamma_5)\not{p}_3(a+b\gamma_5)\Big)\mathrm{Tr}\Big(\not{p}_2(a+b\gamma_5)\not{p}_4(a+b\gamma_5)\Big) \\
&+ \frac{1}{(p_1+p_2)^2-m}\frac{1}{(p_1-p_3)^2-m}\mathrm{Tr}\Big(\not{p}_2(a+b\gamma_5)\not{p}_1(a+b\gamma_5)\not{p}_3(a+b\gamma_5)\not{p}_4(a+b\gamma_5)\Big) \\
&+ \frac{1}{(p_1+p_2)^2-m}\frac{1}{(p_1-p_3)^2-m}\mathrm{Tr}\Big(\not{p}_1(a+b\gamma_5)\not{p}_2(a+b\gamma_5)\not{p}_4(a+b\gamma_5)\not{p}_3(a+b\gamma_5)\Big)\Bigg].
\end{aligned} \tag{30}$$

In the first two lines of Eq. (30) the terms proportional to a^2 don't have any γ_5, the terms in ab cancel due to the property

$$\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\gamma_5\right) = 0\,, \tag{31}$$

while in the term proportional to b^2 we have a structure of the form

$$\mathrm{Tr}\left(\gamma_\mu\gamma_5\gamma_\nu\gamma_5\right) = -\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\right)\,. \tag{32}$$

Coming now to the last two lines, we note that the traces are of the form $\mathrm{Tr}\left(\gamma^\mu(a+b\gamma_5)\gamma^\nu(a+b\gamma_5)\gamma^\rho(a+b\gamma_5)\gamma^\sigma(a+b\gamma_5)\right)$. Consider terms with one single γ_5: there are four, according to the four possible positions of the vertex with γ_5. They cancel in pairs, because

$$\mathrm{Tr}\left(\gamma^\mu\gamma_5\gamma^\nu\gamma^\rho\gamma^\sigma\right) = -\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma_5\gamma^\rho\gamma^\sigma\right). \tag{33}$$

The same applies to terms with three γ_5 matrices: there are four, according to the four possible positions of the vertex without γ_5, and they cancel in pairs, because

$$\mathrm{Tr}\left(\gamma^\mu\gamma_5\gamma^\nu\gamma^\rho\gamma_5\gamma^\sigma\gamma_5\right) = -\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma_5\gamma^\rho\gamma_5\gamma^\sigma\gamma_5\right). \tag{34}$$

This proves that all terms with an odd number of γ_5 matrices cancel or vanish.

The third line of Eq. (30) thus reduces to

$$\begin{aligned}
&\mathrm{Tr}\Big(a^4\not{p}_2\not{p}_1\not{p}_3\not{p}_4 + a^2b^2\not{p}_2\not{p}_1\not{p}_3\gamma_5\not{p}_4\gamma_5 + a^2b^2\not{p}_2\not{p}_1\gamma_5\not{p}_3\not{p}_4\gamma_5 + a^2b^2\not{p}_2\not{p}_1\gamma_5\not{p}_3\gamma_5\not{p}_4 \\
&\quad + a^2b^2\not{p}_2\gamma_5\not{p}_1\not{p}_3\not{p}_4\gamma_5 + a^2b^2\not{p}_2\gamma_5\not{p}_1\not{p}_3\gamma_5\not{p}_4 + a^2b^2\not{p}_2\gamma_5\not{p}_1\gamma_5\not{p}_3\not{p}_4 + b^4\not{p}_2\gamma_5\not{p}_1\gamma_5\not{p}_3\gamma_5\not{p}_4\gamma_5\Big)
\end{aligned} \tag{35}$$

$$\begin{aligned}
&= \mathrm{Tr}\Big(a^4\not{p}_2\not{p}_1\not{p}_3\not{p}_4 + a^2b^2\not{p}_2\not{p}_1\not{p}_3\gamma_5\not{p}_4\gamma_5 + a^2b^2\not{p}_2\gamma_5\not{p}_1\gamma_5\not{p}_3\not{p}_4 + b^4\not{p}_2\gamma_5\not{p}_1\gamma_5\not{p}_3\gamma_5\not{p}_4\gamma_5\Big) \\
&= (a^4+b^4-2a^2b^2)\mathrm{Tr}\left(\not{p}_2\not{p}_1\not{p}_3\not{p}_4\right)
\end{aligned} \tag{36}$$

The fourth line of Eq. (30) is obtained from the third one exchanging $p_1 \leftrightarrow p_2$ and $p_3 \leftrightarrow p_4$, so we finally get

$$\begin{aligned}
\overline{\sum}|\mathcal{M}|^2 = \frac{g^2}{4}(a^2-b^2)^2\Bigg[&\frac{\mathrm{Tr}\left(\not{p}_1\not{p}_2\right)\mathrm{Tr}\left(\not{p}_3\not{p}_4\right)}{((p_1+p_2)^2-m)^2} + \frac{\mathrm{Tr}\left(\not{p}_1\not{p}_3\right)\mathrm{Tr}\left(\not{p}_2\not{p}_4\right)}{((p_1-p_3)^2-m)^2} \\
&- \frac{\mathrm{Tr}\left(\not{p}_2\not{p}_1\not{p}_3\not{p}_4\right)+\mathrm{Tr}\left(\not{p}_1\not{p}_2\not{p}_4\not{p}_3\right)}{((p_1+p_2)^2-m)((p_1-p_3)^2-m)}\Bigg].
\end{aligned} \tag{37}$$

6. Using the trace properties of the gamma matrices we find

$$\begin{aligned}
&\overline{\sum}|\mathcal{M}|^2 = \\
&8\frac{g^2}{4}(a-b)^2(a+b)^2\left[2\frac{(p_1\cdot p_2)(p_3\cdot p_4)}{((p_1+p_2)^2-m)^2} + 2\frac{(p_1\cdot p_3)(p_2\cdot p_4)}{((p_1-p_3)^2-m)^2} + \frac{(p_1\cdot p_4)(p_2\cdot p_3)-(p_1\cdot p_3)(p_2\cdot p_4)-(p_1\cdot p_2)(p_3\cdot p_4)}{((p_1+p_2)^2-m)((p_1-p_3)^2-m)}\right]
\end{aligned} \tag{38}$$

7. Defining Mandelstam variables as

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = 2(p_1 \cdot p_2) = 2(p_3 \cdot p_4)\,, \tag{39}$$

$$t = (p_1 - p_3)^2 = (p_4 - p_2)^2 = -2(p_1 \cdot p_3) = -2(p_2 \cdot p_4)\,, \tag{40}$$

$$u = (p_1 - p_4)^2 = (p_3 - p_2)^2 = -2(p_1 \cdot p_4) = -2(p_2 \cdot p_3)\,, \tag{41}$$

Eq. (38) can be rewritten as

$$\overline{\sum}|\mathcal{M}|^2 = 2g^2(a-b)^2(a+b)^2\left[\frac{s^2}{2(s-m)^2} + \frac{t^2}{2(t-m)^2} + \frac{u^2 - t^2 - s^2}{4(s-m)(t-m)}\right] \tag{42}$$

$$= g^2(a-b)^2(a+b)^2\left[\frac{s^2}{(s-m)^2} - \frac{t^2}{(t-m)^2} + \frac{st}{(s-m)(t-m)}\right], \tag{43}$$

where in the last step we used the property $s + t + u = 0$, since the fermions are massless.

8. We have to compute the unpolarized modulus squared of the amplitude in two cases:

- $p_3 = p_1$ and $p_4 = p_2$: In this case we have that the Mandelstam variables become

$$t = 0, \quad u = (p_1 - p_2)^2 = -2p_1 \cdot p_2 = -s\,. \tag{44}$$

  Eq. (43) becomes

$$\overline{\sum}|\mathcal{M}|^2 = g^2(a-b)^2(a+b)^2\frac{s^2}{(s-m)^2}\,. \tag{45}$$

- $p_3 = p_2$ and $p_4 = p_3$: In this case we have that the Mandelstam variables become

$$t = -s, \quad u = 0\,. \tag{46}$$

  Eq. (43) becomes

$$\overline{\sum}|\mathcal{M}|^2 = g^2(a-b)^2(a+b)^2\left[\frac{s^2}{(s-m)^2} + \frac{s^2}{(s-m)^2} + \frac{s^2}{(s-m)(s+m)}\right]. \tag{47}$$

Therefore, we find that

$$A_{FB}(p_1, p_2) = A(p_1, p_2; p_1, p_2) - A(p_1, p_2; p_2, p_1) = g^2(a-b)^2(a+b)^2\left[\frac{s^2}{(s-m)^2} + \frac{s^2}{(s-m)(s+m)}\right]. \tag{48}$$