Quantum Field Theory I: solution

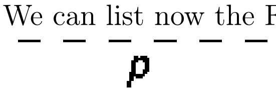
June 30, 2017

1. We start by determining the dimension of the coupling constant g for the Lagrangian

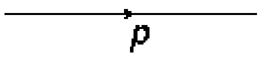
$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi\right) + \bar{\psi}\left(i\not{\partial} - m\right)\psi + g\bar{\psi}\gamma^\mu\gamma^5\psi\partial_\mu\phi. \tag{1}$$

By looking at the free terms of the lagrangian we determine the dimension of ϕ, $\dim(\phi) = 1$ and of $\psi, \bar{\psi}$, $\dim\left(\psi, \bar{\psi}\right) = \frac{3}{2}$. Therefore the dimension of g is equal to $\left[\frac{1}{m}\right]$, the inverse of a mass.

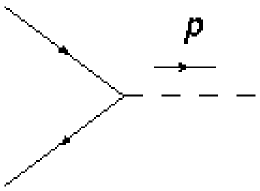
We can list now the Feynman rules for this theory:



$$\frac{i}{p^2 + i\epsilon} \tag{2}$$



$$\frac{i\left(\not{p} + m\right)}{p^2 - m^2 + i\epsilon} \tag{3}$$



$$-g\not{p}\gamma^5 \tag{4}$$

where in the vertex we consider the momentum p flowing outwards.

2. The transition matrix e;ement, at first order in the coupling, for the process

$$f\left(p_1\right) + \bar{f}\left(p_2\right) \to \phi\left(p_3\right) + \phi\left(p_4\right) \tag{5}$$

is given by the two diagrams in Fig. 2. The two corresponding matrix elements are

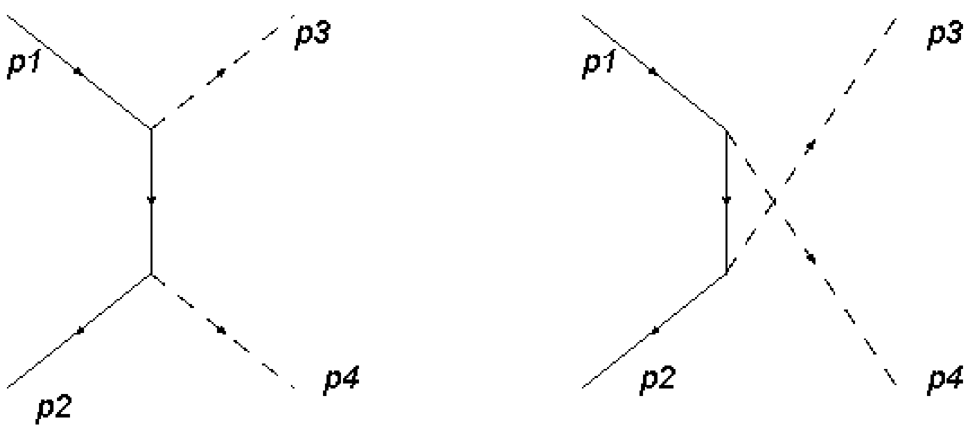


Figura 1: Diagrams contributing to process Eq. (5)

respectively:

$$i\mathcal{M}_t = ig^2 \bar{v}(p_2) \not{p}_4 \gamma^5 \frac{\left(\not{p}_1 - \not{p}_3 + m\right)}{t - m^2} \not{p}_3 \gamma^5 u(p_1), \tag{6}$$

$$i\mathcal{M}_u = ig^2 \bar{v}(p_2) \not{p}_3 \gamma^5 \frac{\left(\not{p}_1 - \not{p}_4 + m\right)}{u - m^2} \not{p}_4 \gamma^5 u(p_1). \tag{7}$$

These can be simplified using the Dirac equation and the commutator suggested in the assignment

$$\not{p}\not{q} = 2(p \cdot q) - \not{q}\not{p}. \tag{8}$$

We obtain:

$$\begin{aligned}
i\mathcal{M}_t &= ig^2 \bar{v}(p_2) \not{p}_4 \gamma^5 \frac{\not{p}_1 - \not{p}_3 + m}{t - m^2} \not{p}_3 \gamma^5 u(p_1) \\
&= ig^2 \bar{v}(p_2) \not{p}_4 \gamma^5 \frac{2(p_1 \cdot p_3) + 2m\not{p}_3}{t - m^2} \gamma^5 u(p_1) \\
&= -ig^2 \bar{v}(p_2) \not{p}_4 \left[1 + \frac{2m\not{p}_3}{t - m^2}\right] u(p_1) && (9) \\
i\mathcal{M}_u &= ig^2 \bar{v}(p_2) \not{p}_3 \gamma^5 \frac{\not{p}_1 - \not{p}_4 + m}{u - m^2} \not{p}_4 \gamma^5 u(p_1) \\
&= ig^2 \bar{v}(p_2) \not{p}_3 \gamma^5 \frac{2(p_1 \cdot p_4) + 2m\not{p}_4}{u - m^2} \gamma^5 u(p_1) && (10) \\
&= -ig^2 \bar{v}(p_2) \not{p}_3 \left[1 + \frac{2m\not{p}_4}{u - m^2}\right] u(p_1) && (11)
\end{aligned}$$

where have introduced the Mandelstam invariants

$$t - m^2 = -2(p_1 \cdot p_3) \qquad u - m^2 = -2(p_1 \cdot p_4). \tag{12}$$

The total amplitude is

$$i\mathcal{M} = -ig^2 \bar{v}(p_2) \left[\left(\not{p}_3 + \not{p}_4\right) + 2m\left(\frac{\not{p}_4\not{p}_3}{t - m^2} + \frac{\not{p}_3\not{p}_4}{u - m^2}\right)\right] u(p_1) \tag{13}$$

which can be further simplified by using the fact that

$$\bar{v}(p_2)\left[\not{p}_3 + \not{p}_4\right] u(p_1) = \bar{v}\left[\not{p}_2 + \not{p}_1\right] u(p_1) = \bar{v}(p_2)[m - m] u(p_1) = 0. \tag{14}$$

We finally obtain the result quoted in the text:

$$i\mathcal{M} = -2mig^2 \bar{v}(p_2) \left[\frac{\not{p}_4\not{p}_3}{t - m^2} + \frac{\not{p}_3\not{p}_4}{u - m^2}\right] u(p_1). \tag{15}$$

3. The sum over final polarizations and the average over initial polarizations of the squared modulus of the matrix element Eq. (15) is

$$
\begin{aligned}
\frac{1}{4}\sum_s |\mathcal{M}|^2 &= m^2 g^4 \mathrm{Tr}\left(\left(\not{p}_1 + m\right)\left[\frac{\not{p}_3\not{p}_4}{t-m^2} + \frac{\not{p}_4\not{p}_3}{u-m^2}\right]\left(\not{p}_2 - m\right)\left[\frac{\not{p}_4\not{p}_3}{t-m^2} + \frac{\not{p}_3\not{p}_4}{u-m^2}\right]\right) \\
&= m^2 g^4 \Bigg\{ \frac{1}{(t-m^2)^2}\mathrm{Tr}\left(\left(\not{p}_1 + m\right)\not{p}_3\not{p}_4\left(\not{p}_2 - m\right)\not{p}_4\not{p}_3\right) \\
&\quad + \frac{1}{(u-m^2)^2}\mathrm{Tr}\left(\left(\not{p}_1 + m\right)\not{p}_4\not{p}_3\left(\not{p}_2 - m\right)\not{p}_3\not{p}_4\right) \\
&\quad + \frac{1}{(u-m^2)(t-m^2)}\Big[\mathrm{Tr}\left(\left(\not{p}_1 + m\right)\not{p}_4\not{p}_3\left(\not{p}_2 - m\right)\not{p}_4\not{p}_3\right) \\
&\qquad + \mathrm{Tr}\left(\left(\not{p}_1 + m\right)\not{p}_3\not{p}_4\left(\not{p}_2 - m\right)\not{p}_3\not{p}_4\right)\Big]\Bigg\} \\
&= m^2 g^4 \Bigg\{ \frac{1}{(t-m^2)^2}\left[\mathrm{Tr}\left(\not{p}_1\not{p}_3\not{p}_4\not{p}_2\not{p}_4\not{p}_3\right)\right] + \frac{1}{(u-m^2)^2}\left[\mathrm{Tr}\left(\not{p}_1\not{p}_4\not{p}_3\not{p}_2\not{p}_3\not{p}_4\right)\right] \\
&\quad + \frac{1}{(u-m^2)(t-m^2)}\Big[\left(\mathrm{Tr}\left(\not{p}_1\not{p}_4\not{p}_3\not{p}_2\not{p}_4\not{p}_3\right) - m^2\mathrm{Tr}\left(\not{p}_4\not{p}_3\not{p}_4\not{p}_3\right)\right) \\
&\qquad + \left(\mathrm{Tr}\left(\not{p}_1\not{p}_3\not{p}_4\not{p}_2\not{p}_3\not{p}_4\right) - m^2\mathrm{Tr}\left(\not{p}_3\not{p}_4\not{p}_3\not{p}_4\right)\right)\Big]\Bigg\} \\
&= m^2 g^4 \Bigg\{ \frac{1}{(t-m^2)^2}\left[2\left(p_2\cdot p_4\right)\mathrm{Tr}\left(\not{p}_1\not{p}_3\not{p}_4\not{p}_3\right)\right] + \frac{1}{(u-m^2)^2}\left[2\left(p_2\cdot p_3\right)\mathrm{Tr}\left(\not{p}_1\not{p}_4\not{p}_3\not{p}_4\right)\right] \\
&\quad + \frac{1}{(u-m^2)(t-m^2)}\Big[\left(-\mathrm{Tr}\left(\not{p}_1\not{p}_4\not{p}_3\not{p}_4\not{p}_2\not{p}_3\right) - 2m^2\left(p_3\cdot p_4\right)\mathrm{Tr}\left(\not{p}_4\not{p}_3\right)\right) \\
&\qquad + \left(-\mathrm{Tr}\left(\not{p}_1\not{p}_3\not{p}_4\not{p}_3\not{p}_2\not{p}_4\right) - 2m^2\left(p_3\cdot p_4\right)\mathrm{Tr}\left(\not{p}_3\not{p}_4\right)\right)\Big]\Bigg\} \\
&= m^2 g^4 \Bigg\{ \frac{1}{(t-m^2)^2}\left[16\left(p_2\cdot p_4\right)\left(p_3\cdot p_4\right)\left(p_1\cdot p_3\right)\right] + \frac{1}{(u-m^2)^2}\left[16\left(p_2\cdot p_3\right)\left(p_3\cdot p_4\right)\left(p_1\cdot p_4\right)\right] \\
&\quad + \frac{1}{(u-m^2)(t-m^2)}\Big[\left(-2\left(p_3\cdot p_4\right)\mathrm{Tr}\left(\not{p}_1\not{p}_4\not{p}_2\not{p}_3\right) - 8m^2\left(p_3\cdot p_4\right)^2\right) \\
&\qquad + \left(-2\left(p_3\cdot p_4\right)\mathrm{Tr}\left(\not{p}_1\not{p}_3\not{p}_2\not{p}_4\right) - 8m^2\left(p_3\cdot p_4\right)^2\right)\Big]\Bigg\} \\
&= 16 m^2 g^4 \Bigg\{ \frac{1}{(t-m^2)^2}\left[\left(p_2\cdot p_4\right)\left(p_3\cdot p_4\right)\left(p_1\cdot p_3\right)\right] + \frac{1}{(u-m^2)^2}\left[\left(p_2\cdot p_3\right)\left(p_3\cdot p_4\right)\left(p_1\cdot p_4\right)\right] \\
&\quad - \frac{1}{(u-m^2)(t-m^2)}\Big[\left(p_3\cdot p_4\right)\left(\left(p_1\cdot p_4\right)\left(p_2\cdot p_3\right) - \left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right) + \left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)\right) \\
&\qquad + m^2\left(p_3\cdot p_4\right)^2\Big]\Bigg\}. \qquad (16)
\end{aligned}
$$

The final result Eq. (16) can be written in terms of Mandelstam invariants using

$$
\left(p_1\cdot p_2\right) = \frac{s-2m^2}{2} \qquad \left(p_3\cdot p_4\right) = \frac{s}{2} \qquad (17)
$$

$$
\left(p_1\cdot p_3\right) = \left(p_2\cdot p_4\right) = \frac{m^2-t}{2} \qquad (18)
$$

$$
\left(p_1\cdot p_4\right) = \left(p_2\cdot p_3\right) = \frac{m^2-u}{2}. \qquad (19)
$$

We get

$$
\begin{aligned}
\frac{1}{4}\sum_s |\mathcal{M}|^2 &= 16m^2g^4\Bigg\{\frac{1}{(t-m^2)^2}\left[(p_2\cdot p_4)(p_3\cdot p_4)(p_1\cdot p_3)\right]+\frac{1}{(u-m^2)^2}\left[(p_2\cdot p_3)(p_3\cdot p_4)(p_1\cdot p_4)\right] \\
&\quad -\frac{1}{(u-m^2)(t-m^2)}\Big[(p_3\cdot p_4)\left((p_1\cdot p_4)(p_2\cdot p_3)-(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_3)(p_2\cdot p_4)\right) \\
&\qquad +m^2(p_3\cdot p_4)^2\Big]\Bigg\} \\
&= 16m^2g^4\Bigg\{\frac{1}{(t-m^2)}\left[\frac{s}{2}\frac{(t-m^2)^2}{4}\right]+\frac{1}{(u-m^2)}\left[\frac{s}{2}\frac{(u-m^2)^2}{4}\right] \\
&\quad -\frac{1}{(u-m^2)(t-m^2)}\left[\frac{s}{2}\left(\frac{(m^2-u)^2}{4}-\frac{s}{2}\frac{(s-2m^2)}{2}+\frac{(m^2-t)^2}{4}\right)+m^2\frac{s^2}{4}\right]\Bigg\} \\
&= 2sm^2g^4\left\{\frac{s^2-(t-u)^2-4m^2s}{(u-m^2)(t-m^2)}\right\} \qquad (20)
\end{aligned}
$$

This result can be rewritten in various equivalent forms. A particularly simple one is

$$
\frac{1}{4}\sum_s |\mathcal{M}|^2 = 8s\,m^2g^4\left\{1-\frac{m^2s}{(u-m^2)(t-m^2)}\right\} \qquad (21)
$$

where we used the fact that

$$
s^2-(u-t)^2=(s+t-u)(s+u-t)=\left(2m^2-2u\right)\left(2m^2-2t\right)=4\left(t-m^2\right)\left(u-m^2\right). \qquad (22)
$$

4. In the center-of-mass frame the kinematics can be written as

$$
\begin{aligned}
p_1 &= \left(\sqrt{p^2+m^2},0,0,p\right), && (23)\\
p_2 &= \left(\sqrt{p^2+m^2},0,0,-p\right), && (24)\\
p_3 &= \left(p',0,0,p'\right), && (25)\\
p_4 &= \left(p',0,0,-p'\right) && (26)
\end{aligned}
$$

with

$$
p=\frac{\sqrt{s}}{2}\beta \qquad (27)
$$

and

$$
\beta=\sqrt{1-\frac{4m^2}{s}}. \qquad (28)
$$

The phase-space for the process Eq. (5) is then

$$
d\Phi=\frac{dp'd\cos\theta d\phi}{16\pi^2}\delta\left(\sqrt{s}-2p'\right)=\frac{d\cos\theta}{16\pi} \qquad (29)
$$

with the constraint $p'=\frac{\sqrt{s}}{2}$.

In order to obtain the differential cross section we first rewrite the square amplitude in terms of center-of-mass variables. Specifically, we use

$$
\begin{aligned}
t-m^2 &= -2(p_1\cdot p_3) = -\frac{s}{2}(1-\beta\cos\theta) && (30)\\
u-m^2 &= -2(p_1\cdot p_3) = -\frac{s}{2}(1+\beta\cos\theta) && (31)
\end{aligned}
$$

to rewrite Eq. (20) as:

$$\frac{1}{4}\sum_s |\mathcal{M}|^2 = 8sm^2g^4\left\{1-\frac{1-\beta^2}{1-\beta^2\cos^2\theta}\right\} \tag{32}$$

with .

The flux factor is

$$4\sqrt{(p_1\cdot p_2)^2-m^4}=2s\beta \tag{33}$$

and the differential cross-section is

$$\frac{d\sigma}{d\cos\theta}=\frac{m^2g^4}{4\pi\beta}\left\{1-\frac{1-\beta^2}{1-\beta^2\cos^2\theta}\right\}. \tag{34}$$

5. The theory is not renormalizable because the coupling g has negative dimensions of mass. A one-loop correction to the interaction (vertex correction) is proportional to g^3 and thus its renormalization requires in general the inclusion of all interactions with couplings with dimension $\left[\frac{1}{m^3}\right]$.

   We seek all Lorentz invariant terms with dimension five, i.e. $\left[m^5\right]$. Lorentz invariant interactions must be built out of fermion bilinears, with dimension $\left[m^3\right]$, the scalar field, with dimension $[m]$, and derivatives, with dimension $[m]$. Two fermion bilinears have dimension six ($\left[m^6\right]$) which requires a coupling with dimension $\left[m^2\right]$ or higher, so at most one fermion bilinear is allowed.

   Therefore, only interactions with either zero or one fermion bilinears are allowed.

   With one fermion bilinear, we can have either a scalar or a pseudoscalar and two powers of the scalar field, or a vector or a pseudovector contracted with a derivative and one power of the scalar field. The latter contribution is the only one which is already included in the given Lagrangian.

   With zero fermion bilinears we can have only powers of the scalar field and derivatives. A Lorentz-invariant combination of derivatives of the scalar field can be built with an even number of derivatives. Therefore, the possible dimension five interactions have either five powers of the field, or three powers and two derivatives, or one power and four derivatives. A term with only one power of the field gives rise to no interaction.

   Summarizing, the most general lagrangian is

$$\mathcal{L} = g_1\bar{\psi}\psi\phi^2 + g_1'\bar{\psi}\gamma_5\psi\phi^2 + g_2\bar{\psi}\gamma^\mu\psi\partial_\mu\phi + g_2'\bar{\psi}\gamma^\mu\gamma_5\psi\partial_\mu\phi \tag{35}$$

$$+\lambda_1\phi^5+\lambda_2\phi\partial_\mu\phi\partial^\mu\phi. \tag{36}$$

   The only interaction included in the given Lagrangian is that proportional to g_2'. All the others would be generated upon renormalization.

6. In the limit $m\to 0$, the differential cross section vanishes. This can be understood by examining the symmetries of the Lagrangian and the associated conserved Noether currents.

   When $m\neq 0$ the Lagrangian is invariant under the $U(1)_V$ transformation

$$\psi\to\psi'=e^{i\alpha}\psi. \tag{37}$$

   The conserved Noether current is the vector current

$$J_V^\mu=\bar{\psi}\gamma^\mu\psi;\qquad\qquad \partial_\mu J_V^\mu=0. \tag{38}$$

   When $m=00$ the Lagrangian is also invariant under the $U(1)_A$ transformation

$$\psi\to\psi'=e^{i\alpha\gamma_5}\psi. \tag{39}$$

The conserved Noether current is the axial current

$$J_A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi; \qquad \partial_m u J_A^\mu = 0. \tag{40}$$

In the latter case, using the conservation law

$$\partial^\mu J_A^\mu = 0 \tag{41}$$

and integrating by parts the interaction we get

$$\mathcal{L}_I = \bar{\psi}\gamma^\mu\gamma^5\psi\partial_\mu\phi = \partial_\mu J_A^\mu\phi = 0 \tag{42}$$

so the interaction term vanishes.

Note that this means that the interaction term proportional to g_2 in Eq. (35) actually vanishes.