QUANTUM FIELD THEORY I

Solution

June 28, 2019

Consider a theory of a Dirac fermion f and a real scalar s, with field ψ and ϕ respectively, given by the following Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{int}}. \tag{1}$$

The Lagrangian of the scalar part is given by

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - M^2\phi^2\right), \tag{2}$$

where the field ϕ is defined as follows

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left\{ a_{\vec{p}}\, \mathrm{e}^{-ip^\mu x_\mu} + a^\dagger_{\vec{p}}\, \mathrm{e}^{ip^\mu x_\mu} \right\}, \tag{3}$$

with the commutation relation:

$$[a_{\vec{p}}, a^\dagger_{\vec{k}}] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{k}). \tag{4}$$

The fermion part is given by

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\partial\!\!\!/ - m)\psi, \tag{5}$$

where the field ψ is defined as follows

$$\psi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_s \left\{ b^s_{\vec{p}} u^s(p)\mathrm{e}^{-ip^\mu x_\mu} + c^{s\dagger}_{\vec{p}} v^s(p)\mathrm{e}^{ip^\mu x_\mu} \right\}, \tag{6}$$

with the following anti-commutation relation:

$$\{b^s_{\vec{p}}, b^{r\dagger}_{\vec{k}}\} = \{c^s_{\vec{p}}, c^{r\dagger}_{\vec{k}}\} = (2\pi)^3\delta^{sr}\delta^{(3)}(\vec{p}-\vec{k}). \tag{7}$$

Finally, the Lagrangian of the interatcion is given by the Yukawa,

$$\mathcal{L}_{\text{int}} = g\bar{\psi}\psi\phi. \tag{8}$$

(1) For this theory, the energy-momentum tensor is given by:

$$\begin{aligned} T^\mu{}_\nu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\nu\psi + \delta^\mu{}_\nu\mathcal{L} &(9)\\ &= \partial^\mu\phi\partial_\nu\phi + i\bar{\psi}\gamma^\mu\partial_\nu\psi - \delta^\mu{}_\nu\mathcal{L}. &(10)\end{aligned}$$

(2) According to Noether's theorem, a conservation law is associated with a symmetry. One can, indeed, straightforwardly check that the Lagrangian is invariant under the following transformation:

$$\psi \longrightarrow \psi' = \mathrm{e}^{-iq}\psi. \tag{11}$$

The associated symmetry transformations form the group $U(1)$. (*Not requested for full grades*)

The corresponding Noether current is

$$\mathcal{J}^{\mu} = \frac{\mathcal{L}_{\text{Dirac}}}{\partial(\partial_{\mu}\psi)}\Delta\psi \qquad \text{with} \qquad \frac{\mathcal{L}_{\text{Dirac}}}{\partial(\partial_{\mu}\psi)} = i\bar{\psi}\gamma^{\mu}, \tag{12}$$

$\Delta\psi$ can be determined from an infinitesimal transformation

$$\psi \longrightarrow \psi + \Delta\psi, \qquad \Delta = -iq\psi. \tag{13}$$

Combining this result to the ones in Eq. (12), we finally have

$$\mathcal{J}^{\mu} = q\bar{\psi}\gamma^{\mu}\psi. \tag{14}$$

(3) Now, let us compute the expression of the conserved charge corresponding to the previous Noether's current and express it in terms of the creation and annihilation operators. The Dirac charge operator is given by

$$\mathcal{Q} = \int \mathrm{d}^3x\, \mathcal{J}^0(x) = q \int \mathrm{d}^3x\, \psi^{\dagger}(x)\psi(x) \tag{15}$$

Expanding the above expression, we have the following

$$\begin{aligned} \mathcal{Q} &= q \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{r,s} \left\{ b_{\vec{p}}^{r\dagger} b_{\vec{p}}^{s} u^{r\dagger}(p) u^{s}(p) + c_{-\vec{p}}^{r} c_{-\vec{p}}^{s\dagger} v^{r\dagger}(-p) v^{s}(-p) \right\} &(16)\\ &= q \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{s} \left(b_{\vec{p}}^{s\dagger} b_{\vec{p}}^{s} + c_{-\vec{p}}^{s} c_{-\vec{p}}^{s\dagger} \right) &(17)\\ &= q \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{s} \left(b_{\vec{p}}^{s\dagger} b_{\vec{p}}^{s} - c_{\vec{p}}^{s\dagger} c_{\vec{p}}^{s} \right). &(18) \end{aligned}$$

In order to get the first line, we integrated over x then over the second momentum. From the first to the second line, we used the fact that $u^{r\dagger}(p)u^s(p) = v^{r\dagger}(p)v^s(p) = 2E_p\delta^{rs}$. Notice that by symmetry $c_{-\vec{p}}^{s} c_{-\vec{p}}^{s\dagger} = c_{\vec{p}}^{s} c_{\vec{p}}^{s\dagger}$. To get to the last line, we wrote $c_{\vec{p}}^{s} c_{\vec{p}}^{s\dagger}$ in terms if its anti-commutation.

(4) Let us now write down the Feynman rules of the theory.

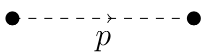
(a) External lines: the Feynman rules for the external lines can be obtained by acting the fields on the initial and final-state particles.

$$\phi|s(p)\rangle = \underset{p}{\dashrightarrow}\bullet = 1 \qquad\qquad \langle s(p)|\phi = \bullet\underset{p}{\dashrightarrow} = 1$$

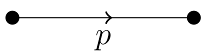
$$\psi|f(p,s)\rangle = \underset{p}{\longrightarrow}\bullet = u^s(p) \qquad\qquad \langle f(p,s)|\bar{\psi} = \bullet\underset{p}{\longrightarrow} = \bar{u}^s(p)$$

$$\bar{\psi}|\bar{f}(p,s)\rangle = \underset{p}{\longleftarrow}\bullet = \bar{v}^s(p) \qquad\qquad \langle \bar{f}(p,s)|\psi = \bullet\underset{p}{\longleftarrow} = v^s(p)$$

(b) The propagators can be computed from the contracted fields.

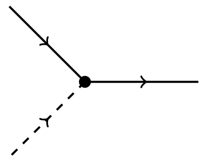
$$\phi(x)\phi(y) =$$ 

$$= \frac{i}{p^2 - M^2 + i\epsilon} \tag{19}$$

$$\psi(x)\psi(y) =$$ 

$$= \frac{i(\not{p} + m)}{p^2 - M^2 + i\epsilon} \tag{20}$$

(c) Finally, the scalar-fermion vertex is given by


$$= -ig \tag{21}$$

It can be seen from the Lagrangian that the coupling g is dimensionless and therefore the theory is renormalizable. Indeed, considering $\hbar = c = 1$, we know that the Lagrangian has dimension $[E]^4$ while ψ and $\bar{\psi}$ have each dimension $[E]^{3/2}$, and ϕ has dimension $[E]$.

(5) In this section, we are going to compute the modulus square of the unpolarized amplitude for the process $fs \to fs$. As shown in Fig. 1, two diagrams contribute to such process.

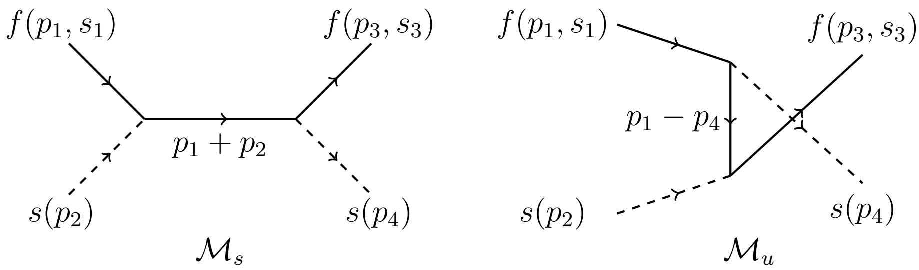


Figure 1: Leading non-vanishing diagrams for $fs \to fs$.

Using the Feynman rules from the previous section, each diagram in the above figure can be mathematically expressed as follows:

$$i\mathcal{M}_s(fs \to fs) = i(ig)^2 \bar{u}(p_3) \left(\frac{\not{p}_1 + \not{p}_2 + m}{(p_1 + p_2)^2 - m} \right) u(p_1) \tag{22}$$

$$i\mathcal{M}_u(fs \to fs) = i(ig)^2 \bar{u}(p_3) \left(\frac{\not{p}_1 - \not{p}_4 + m}{(p_1 - p_4)^2 - m} \right) u(p_1) \tag{23}$$

From these expressions, one can notice that the two amplitudes are related by symmetry–to be precise by a swap of p_2 and $-p_4$. For simplicity, let us define the following quantity:

$$\mathrm{M} = \frac{\not{p}_1 + \not{p}_2 + m}{(p_1 + p_2)^2 - m} + \frac{\not{p}_1 - \not{p}_4 + m}{(p_1 - p_4)^2 - m} \tag{24}$$

$$= \frac{\not{p}_2 + 2m}{s - m^2} - \frac{\not{p}_4 - 2m}{u - m^2}, \tag{25}$$

where in the last line we have introduced the Mandelstam variables. It then follows that the total amplitude can be written as:

$$\mathcal{M}(fs \to fs) = \mathcal{M}_s(fs \to fs) + \mathcal{M}_u(fs \to fs) = -g^2\bar{u}(p_3)\mathrm{M}u(p_1). \tag{26}$$

In order to derive the expression of the amplitude square, one has to compute the complex conjugate of Eq. (26):

$$\begin{aligned}
\mathcal{M}^*(fs \to fs) &= -g^2\left[u(p_3)\right]^T(\gamma^0)^*\mathrm{M}^*u^*(p_1) &(27)\\
&= -g^2u^\dagger(p_1)\mathrm{M}^\dagger\gamma^0u(p_3) &(28)\\
&= -g^2\bar{u}(p_1)\mathrm{M}u(p_3). &(29)
\end{aligned}$$

The square modulus of the amplitude is

$$\begin{aligned}
|\mathcal{M}(fs \to fs)|^2 &= g^4\bar{u}_\alpha(p_3)\mathrm{M}_{\alpha\beta}u_\beta(p_1)\bar{u}_\gamma(p_1)\mathrm{M}_{\gamma\sigma}u_\sigma(p_3) &(30)\\
&= g^4\left[u_\sigma(p_3)\bar{u}_\alpha(p_3)\right]\mathrm{M}_{\alpha\beta}\left[u_\beta(p_1)\bar{u}_\gamma(p_1)\right]\mathrm{M}_{\gamma\sigma} &(31)\\
&= g^4\mathrm{Tr}\left\{\left[u(p_3)\bar{u}(p_3)\right]\mathrm{M}\left[u(p_1)\bar{u}(p_1)\right]\mathrm{M}\right\}. &(32)
\end{aligned}$$

Summing over the polarization of the final state and averaging over the polarization of the initial state we get

$$|\overline{\mathcal{M}}(fs \to fs)|^2 = \frac{1}{2}\sum_{s_1}\sum_{s_3}|\mathcal{M}(fs \to fs)|^2. \tag{33}$$

Expanding the right-hand side of this equation and invoking the spinor completeness relation, it follows that:

$$|\overline{\mathcal{M}}(fs \to fs)|^2 = \frac{g^4}{2}\mathrm{Tr}\left[(\not{p}_3 + m)\,\mathrm{M}\,(\not{p}_1 + m)\,\mathrm{M}\right]. \tag{34}$$

One can notice that the above expression is purely expressed in terms of a trace over gamma matrices. Expanding this expression will give rise to four terms:

$$\begin{aligned}
\xi_{s,s} &= \frac{g^4}{(s-m^2)^2}\tilde{\xi}_{s,s}, & \tilde{\xi}_{s,s} &= \frac{1}{2}\mathrm{Tr}\left[(\not{p}_3+m)(\not{p}_2+2m)(\not{p}_1+m)(\not{p}_2+2m)\right] &(35)\\
\xi_{u,u} &= \frac{g^4}{(u-m^2)^2}\tilde{\xi}_{u,u}, & \tilde{\xi}_{u,u} &= \frac{1}{2}\mathrm{Tr}\left[(\not{p}_3+m)(\not{p}_4-2m)(\not{p}_1+m)(\not{p}_4-2m)\right] &(36)\\
\xi_{s,u} &= -\frac{g^4}{\chi(s,u,m)}\tilde{\xi}_{s,u}, & \tilde{\xi}_{s,u} &= \frac{1}{2}\mathrm{Tr}\left[(\not{p}_3+m)(\not{p}_2+2m)(\not{p}_1+m)(\not{p}_4-2m)\right] &(37)\\
\xi_{u,s} &= -\frac{g^4}{\chi(s,u,m)}\tilde{\xi}_{u,s}, & \tilde{\xi}_{u,s} &= \frac{1}{2}\mathrm{Tr}\left[(\not{p}_3+m)(\not{p}_4-2m)(\not{p}_1+m)(\not{p}_2+2m)\right] &(38)
\end{aligned}$$

where $\chi(s,u,m) = (s-m^2)(u-m^2)$. Since, s- and u-channel are related by symmetry, one only has to compute two of the expressions above, namely $\xi_{s,s}$ and $\xi_{s,u}$, and get the remaining expressions by swapping p_2 and $-p_4$ or s and u.

When expressed in terms of the Mandelstam variables and the masses of the particles, the scalar products can be written as:

$$2(p_1p_3) = 2m^2 - t, \tag{39}$$
$$2(p_2p_4) = 2M^2 - t, \tag{40}$$
$$2(p_1p_2) = 2(p_3p_4) = s - m^2 - M^2, \tag{41}$$
$$2(p_2p_3) = 2(p_1p_4) = m^2 + M^2 - u. \tag{42}$$

Expanding Eq. (35) and (37) and computing the traces one finds,

$$\tilde{\xi}_{s,s} = (9s+u)m^2 - us + 7m^4 - 8m^2M^2 + M^4, \tag{43}$$
$$\tilde{\xi}_{s,u} = -3(u+s)m^2 - us - 9m^4 + 8m^2M^2 + M^4. \tag{44}$$

Recall that $\tilde{\xi}_{u,u} = \tilde{\xi}_{s,s}(s \leftrightarrow u)$ and in the same way $\tilde{\xi}_{s,u} = \tilde{\xi}_{u,s}(s \leftrightarrow u)$.

Combining all these results, we can finally write down the final expression of the amplitude square, given as

$$|\overline{\mathcal{M}}(fs \to fs)|^2 = g^4 \left[\frac{\tilde{\xi}_{s,s}}{(s-m^2)^2} + \frac{\tilde{\xi}_{u,u}}{(u-m^2)^2} - \frac{2\tilde{\xi}_{s,u}}{(s-m^2)(u-m^2)} \right]. \tag{45}$$

(6) In the case where $m = M = 0$ (massless fermions), the *free* theory exhibits the symmetry defined by the transformation

$$\psi_L \longrightarrow \mathrm{e}^{i\theta_L}\psi_L \qquad \text{and} \qquad \psi_R \longrightarrow \mathrm{e}^{i\theta_R}\psi_R, \tag{46}$$

in which ψ_L and ψ_R are rotated by two independent angles θ_L and θ_R respectively. This is called chiral symmetry. However, the coupling term has exactly the same form as a mass term, and thus breaks this symmetry even when the masses vanish, unless the coupling g also vanishes.

Therefore, with $m = M = g = 0$ the group of symmetry is enlarged from $U(1)$ to $U(1) \times U(1)$, but when $g \neq 0$ the internal symmetries of the theory are unchanged even with vanishing masses.

(7) Let us finally compute the trace of the energy-momentum tensor of the theory and show that the energy-momentum tensor is traceless for a particular choice of M, m and g of the theory. Following Eq. (10), the trace of the energy-momentum tensor can be written as

$$T^{\mu}_{\ \mu} = \partial^\mu\phi\partial_\mu\phi + i\bar{\psi}\partial\!\!\!/\psi - 4\mathcal{L} \tag{47}$$
$$= -\partial^\mu\phi\partial_\mu\phi - 3i\bar{\psi}\partial\!\!\!/\psi + 2M^2\phi^2 + 4m\bar{\psi}\psi - 4g\phi\bar{\psi}\psi. \tag{48}$$

From the first to the second line, we just expanded the expression.

The equations of motion for the scalar and fermion fields are respectively given by

$$\partial_\mu \frac{\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial\mathcal{L}}{\partial\phi} \Longrightarrow (\Box + M^2)\phi - g\bar{\psi}\psi = 0 \tag{49}$$
$$\partial_\mu \frac{\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = \frac{\partial\mathcal{L}}{\partial\bar{\psi}} \Longrightarrow (i\partial\!\!\!/ - m + g\phi)\psi = 0. \tag{50}$$

The first term in the second line of Eq. (48) can be simplified further by doing an integration by parts: this corresponds to adding a total derivative, which is allowed because Noether currents are only defined up to a total derivative which does not change the conserved charge. We get

$$
\begin{aligned}
T^{\mu}_{\ \mu} &= \phi\Box\phi - 3i\bar{\psi}\partial\!\!\!/\psi + 2M^2\phi^2 + 4m\bar{\psi}\psi - 4g\phi\bar{\psi}\psi && (51)\\
&= -3g\phi\bar{\psi}\psi + M^2\phi^2 - 3i\bar{\psi}\partial\!\!\!/\psi + 4m\bar{\psi}\psi && (52)\\
&= m\bar{\psi}\psi + M^2\phi^2. && (53)
\end{aligned}
$$

To go from the first to the second line, we used the equation of motion for the scalar field ϕ, while we used the equation of motion for the fermion field $\bar{\psi}$ to go from the second to the third line. From the last line, it can be clearly seen that the energy-momentum is traceless for $m = M = 0$.

So when the masses vanish not only the internal symmetries but also the space-time symmetries are greater. The vanishing of the energy momentum tensor follows from the conservation of the Noether current associated to invariance upon dilatations, i.e. rescaling of all coordinates.