QUANTUM FIELD THEORY I

Solution

June 23, 2021

Consider a theory which involves two Dirac fermions represented by the fields ψ_1 and ψ_2, and a real scalar with field ϕ given by the following Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{int}}. \tag{1}$$

The Lagrangian of the scalar part is given by

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right), \tag{2}$$

where the field ϕ is defined as follows

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left\{a_{\vec{p}}\, \mathrm{e}^{-ip^\mu x_\mu} + a^\dagger_{\vec{p}}\, \mathrm{e}^{ip^\mu x_\mu}\right\}, \tag{3}$$

withe the annihilation and creation operators satisfying the commutation relation:

$$[a_{\vec{p}}, a^\dagger_{\vec{k}}] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{k}). \tag{4}$$

The free fermion part is given by

$$\mathcal{L}_{\text{fermion}} = \sum_{i=1,2}\bar{\psi}_i(i\partial\!\!\!/ - m_i)\psi_i, \tag{5}$$

where a field ψ_i is defined as follows

$$\psi_i(\vec{x},t) = \int \frac{d^3p_i}{(2\pi)^3\sqrt{2E_p}}\sum_s\left\{b^s_{\vec{p}}\, u^s_i(p)\mathrm{e}^{-ip^\mu x_\mu} + c^{s\dagger}_{\vec{p}}\, v^s_i(p)\mathrm{e}^{ip^\mu x_\mu}\right\}, \tag{6}$$

with the annihilation and creation operators satisfying the anticommutation relation:

$$\{b^s_{\vec{p}}, b^{r\dagger}_{\vec{k}}\} = \{c^s_{\vec{p}}, c^{r\dagger}_{\vec{k}}\} = (2\pi)^3\delta^{sr}\delta^{(3)}(\vec{p}-\vec{k}). \tag{7}$$

Finally, the Lagrangian of the interaction is given by:

$$\mathcal{L}_{\text{int}} = ig\sum_{i=1,2}\bar{\psi}_i\phi\psi_i. \tag{8}$$

(1) The energy-momentum tensor for this theory is given by:

$$T^\mu_{\ \nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_1)}\partial_\nu\psi_1 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_2)}\partial_\nu\psi_2 - \delta^\mu_{\ \nu}\mathcal{L} \tag{9}$$

$$= \partial^\mu\phi\partial_\nu\phi + i\bar{\psi}_1\gamma^\mu\partial_\nu\psi_1 + i\bar{\psi}_2\gamma^\mu\partial_\nu\psi_2 - \delta^\mu_{\ \nu}\mathcal{L}. \tag{10}$$

(2) For fermions, the electric charge is always conserved, and according to the Noether's theorem a conservation law is always associated with a symmetry. Let us write down the transformations that leave the Lagrangian invariant and derive the corresponding Noether's currents.

- Conservation of the charge q_1:

$$\psi_1 \longrightarrow \psi_1^{'} = \mathrm{e}^{-iq_1}\psi_1. \tag{11}$$

The corresponding Noether current can be written as:

$$\mathcal{J}_1^\mu = \frac{\partial \mathcal{L}_{\text{fermion}}}{\partial(\partial_\mu \psi_1)}\Delta\psi_1 \quad \text{with} \quad \frac{\partial \mathcal{L}_{\text{fermion}}}{\partial(\partial_\mu \psi_1)} = i\bar{\psi}_1\gamma^\mu \quad \text{and} \quad \Delta\psi_1 = -iq_1\psi_1. \tag{12}$$

Combining the above results yields:

$$\mathcal{J}_1^\mu = q_1\bar{\psi}_1\gamma^\mu\psi_1. \tag{13}$$

- Conservation of the charge q_2:

$$\psi_2 \longrightarrow \psi_2^{'} = \mathrm{e}^{-iq_2}\psi_2. \tag{14}$$

The corresponding Noether current can be written as:

$$\mathcal{J}_2^\mu = \frac{\partial \mathcal{L}_{\text{fermion}}}{\partial(\partial_\mu \psi_2)}\Delta\psi_2 \quad \text{with} \quad \frac{\partial \mathcal{L}_{\text{fermion}}}{\partial(\partial_\mu \psi_2)} = i\bar{\psi}_2\gamma^\mu \quad \text{and} \quad \Delta\psi_2 = -iq_2\psi_2. \tag{15}$$

Similarly, combining the above results yields:

$$\mathcal{J}_2^\mu = q_2\bar{\psi}_2\gamma^\mu\psi_2. \tag{16}$$

The symmetry transformation is given by the $[U(1)]_{q_i}$ group which corresponds to the conservation of charge q_i. Because there are two conserved charges, the global symmetry group is therefore given by $[U(1)]_{q_1} \otimes [U(1)]_{q_2}$.

(3) We expect that at the quantum level all classically conserved charge become operators that commute with the Himiltonian. We check this by explicit calculation.

The charge operator for each species of fermions is defined in terms of the Noether's current as

$$Q_i = \int \mathrm{d}^3x\, \mathcal{J}_i^0(x) = q_i \int \mathrm{d}^3x\, \psi_i^\dagger(x)\psi_i(x) \tag{17}$$

which can be expanded using the definition of the Dirac field to arrive to the following expressions:

$$Q_i = q_i \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} \sum_{r,s} \left(b_{\vec{p}}^{r\dagger} b_{\vec{p}}^s u_i^{r\dagger}(p)u^s(p) + c_{-\vec{p}}^r c_{-\vec{p}}^{s\dagger} v_i^{r\dagger}(-p)v^s(-p) \right) \tag{18}$$

$$= q_i \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{s} \left(b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s + c_{\vec{p}}^s c_{\vec{p}}^{s\dagger} \right) \tag{19}$$

The first line is derived by first integrating over x and then over the second momentum. From the first to the second line, we first used the orthogonal properties of the spinors $u_i^{r\dagger}(p)u^s(p) = v_i^{r\dagger}(p)v^s(p) = 2E_p\delta^{r,s}$, then by symmetry replaced $c_{-\vec{p}}^s c_{-\vec{p}}^{s\dagger}$ with just $c_{\vec{p}}^s c_{\vec{p}}^{s\dagger}$.

The Hamiltonian density is, instead, derived from the energy-momentum tensor through the relation $\mathcal{H} = T^{00}$ from which the Hamiltonian operator is found to be $H = \int \mathrm{d}^3x\, \mathcal{H}$. The contribution to the Hamiltonian operator can be organized in terms of the species of fermions. For a fermion i, the Hamiltonian operator H_i can be derived from Eq. (10) using the Dirac equation:

$$H_i = i \int \mathrm{d}^3x\, \psi_i^\dagger \partial_t \psi_i. \tag{20}$$

By performing the exact same calculation as for the charge operator, we find:

$$H_i = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p \sum_s \left(b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s - c_{\vec{p}}^s c_{\vec{p}}^{s\dagger} \right). \tag{21}$$

Computing the commutation relation $[H_i, Q_i]$ involves the computation of the following terms $\left[b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s, b_{\vec{k}}^{r\dagger} b_{\vec{k}}^r\right]$ and $\left[c_{\vec{p}}^s c_{\vec{p}}^{s\dagger}, c_{\vec{k}}^r c_{\vec{k}}^{r\dagger}\right]$ which can be written in terms of anti-commutation relations. For instance,

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p \left[b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s, b_{\vec{k}}^{r\dagger} b_{\vec{k}}^r\right] = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p \left(b_{\vec{p}}^{s\dagger} \left\{ b_{\vec{p}}^s, b_{\vec{k}}^{r\dagger} \right\} b_{\vec{k}}^r - b_{\vec{k}}^{r\dagger} \left\{ b_{\vec{p}}^{s\dagger}, b_{\vec{k}}^r \right\} b_{\vec{p}}^s \right) = 0. \tag{22}$$

Similarly one can show that the commutator of the term with c-operators also vanishes. It follows that the charge operators Q_i commute with the Hamiltonian, as we set out to prove:

$$[Q_i, H] = 0. \tag{23}$$

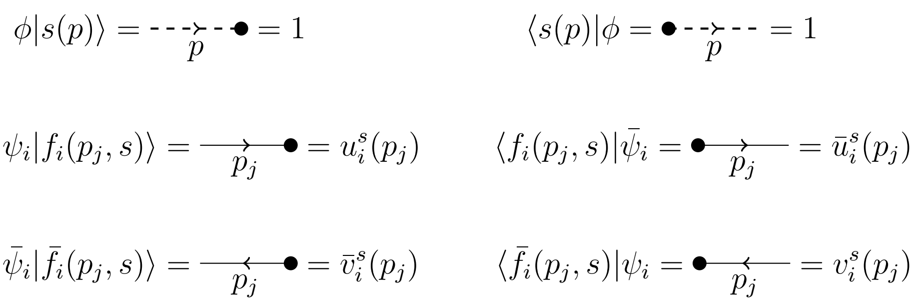
Because the creation and annihilation operators corresponding to different Fermion species anticommute, and exploiting again the bilinearity in order to express the commutator in terms of anticommutators as in Eq. (22) it is easy to see that

$$[Q_1, Q_2] = 0. \tag{24}$$

Hence we conclude that the maximal set of operators that commute with the Hamiltonian is given by Q_1 and Q_2, and that they commute with each other. So also at the quantum level the symmetry group is $[U(1)]_{q_1} \otimes [U(1)]_{q_2}$.

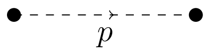
(4) Let us now write down the Feynman rules of the theory.

(*a*) External lines: the Feynman rules for the external lines can be obtained by acting the fields on the initial and final-state particles.

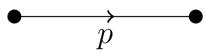


The index on the fields and spinors denote the species of fermion.

(b) The propagators can be computed from the contracted fields.

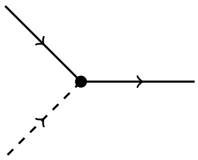


$$\phi(x)\phi(y) = \quad = \frac{i}{p - m_\phi^2 + i\epsilon} \tag{25}$$



$$\psi_i(x)\psi_i(y) = \quad = \frac{i(\not{p}_j + m)}{p_j - m_i^2 + i\epsilon} \tag{26}$$

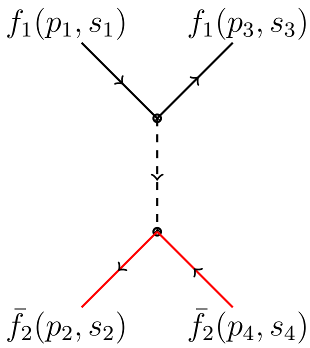
(c) Finally, the scalar-fermion vertex is given by



$$= -g \tag{27}$$

(5) In this section, we are going to compute the modulus square of the unpolarized amplitude for the elastic scattering $f_1\bar{f}_2 \to f_1\bar{f}_2$. As shown in Eq. (28), only one single diagram contributes to this process.

$$i\mathcal{M}_t(f_1\bar{f}_2 \to f_1\bar{f}_2) \equiv \tag{28}$$



Here, we used the black line to denote a fermion of type 1 and the red line to denote a fermion of type 2. Using the Feynman rules from the previous section, we can write down the expression of the scattering amplitude given by Eq. (28):

$$i\mathcal{M}_t(f_1\bar{f}_2 \to f_1\bar{f}_2) = ig^2 \bar{u}_1^s(p_3) u_1^s(p_1) \frac{1}{t - m_\phi^2} v_2^s(p_4) \bar{v}_2^s(p_2), \tag{29}$$

where we used the definition of the Mandelstam variable $t = (p_1 - p_3)^2$. Multiplying Eq. (29) by its hermitian conjugate, summing over the polarization of the final states and averaging over the polarization of the initial states, we get

$$|\bar{\mathcal{M}}_t|^2 = \frac{1}{4} \sum_{s_i} |\mathcal{M}_t(f_1\bar{f}_2 \to f_1\bar{f}_2)|^2 \tag{30}$$

$$= \frac{g^4}{4(t - m_\phi)^2} \mathrm{Tr}\left[(\not{p}_1 + m_1)(\not{p}_3 + m_1)\right] \mathrm{Tr}\left[(\not{p}_2 + m_2)(\not{p}_4 + m_2)\right], \tag{31}$$

where in order to go from the first to the second line we used the completeness relations of the spinors $\sum_s \bar{u}_i^s(p_i) u_i^s(p_i) = \not{p}_i + m_i$ and $\sum_s \bar{v}_i^s(p_i) v_i^s(p_i) = \not{p}_i - m_i$.

The above expression can be simplified further using the properties of the trace of gamma matrices:

$$|\bar{\mathcal{M}}_t|^2 = \frac{4g^4}{(t-m_\phi)^2}\left[(p_1p_3)+m_1^2\right]\left[(p_2p_4)+m_2^2\right] \tag{32}$$

$$= \frac{g^4}{(t-m_\phi)^2}\left(t-4m_1^2\right)\left(t-4m_2^2\right), \tag{33}$$

where in order to arrive to the last line we used again the definition of the Mandelstam variable $(p_1p_3) = m_1^2 - t/2$ and $(p_2p_4) = m_2^2 - t/2$.

(6) Let us now compute the amplitude derived in the previous section in the limit $m_\phi \to \infty$. Expanding Eq. (33) as a series in $1/m_\phi$ leads to the following expression:

$$|\bar{\mathcal{M}}_t(m_\phi \to \infty)|^2 = \left(\frac{g}{m_\phi}\right)^4 \left(t-4m_1^2\right)\left(t-4m_2^2\right) + \mathcal{O}\left(\frac{1}{m_\phi^5}\right). \tag{34}$$

As shown in Fig. 1, this corresponds to a diagram where the scalar propagator shrinks leading to a 4-Fermi vertex amplitude whose theory is governed by the following Lagrangian:

$$\mathcal{L}' = \mathcal{L}(m_\phi \to \infty) = \sum_{i=1,2} \bar{\psi}_i(i\partial\!\!\!/ - m_i)\psi_i + G\left(\bar{\psi}\psi\right)^2, \tag{35}$$

where the coupling G is related to g via the relation $G = g^2/m_\phi^2$.

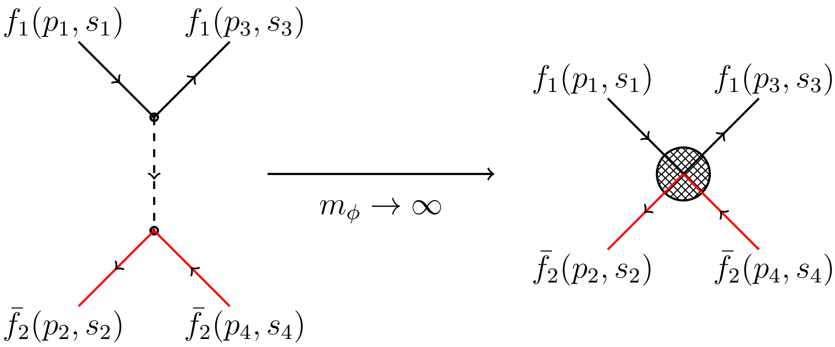


Figure 1: Large-m_ϕ limit diagram for $f_1\bar{f}_2 \to f_1\bar{f}_2$.

Considering $\hbar = c = 1$, we know that the Lagrangian has dimension $[E]^4$ while ψ and $\bar{\psi}$ have each dimension $[E]^{3/2}$, and ϕ has dimension $[E]$. It can be read from the Lagrangian $\mathcal{L}$ that the coupling g is dimensionless and therefore the theory is renormalizable. However, the coupling G in $\mathcal{L}'$ has dimension $[E]^{-2}$ and therefore the theory is not renormalizable.

(7) In the case $m_1 = m_2 = m$, there is a larger symmetry since we can mix ψ_1 and ψ_2 by introducing $\Psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$ such that the Lagrangian in Eq. (2) can be written as:

$$\tilde{\mathcal{L}} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) + \bar{\Psi}(i\partial\!\!\!/ - m + ig\phi)\Psi. \tag{36}$$

We can see that the above Lagrangian is invariant under the transformation $\Psi \rightarrow U\Psi$ where U is a unitary two-by-two matrix that satisfies $UU^\dagger = I$. The complex matrices that satisfy this condition form a group $U(2) = SU(2) \otimes U(1)$ where the symmetry group $U(1)$ corresponds to the conservation of the total charge while $SU(2)$ corresponds to the conservation of the relative charge.

Defining $U = \exp(i\theta_a \sigma_a)$, with σ_a denoting the Pauli matrices which are the generators of the group $SU(2)$, the Noether currents are found to be $\mathcal{J}_a^\mu = \bar{\Psi}\gamma^\mu \sigma_a \Psi$ for the SU(2) transformation and $\mathcal{J}_a^\mu = \bar{\Psi}\gamma^\mu \Psi$ for U(1). From these we get charges Q_a and Q respectively, where at the quantum level Ψ is now a (doublet) Fermion field operator, and

$$\pi = \partial \tilde{\mathcal{L}} / \partial \left(\partial_t \Psi\right) = i\Psi^\dagger \tag{37}$$

is the corresponding canonical momentum operator that satisfies the anticommutation relation

$$\{\Psi(x)_i, \pi_j(y)\} = i\delta_{ij}\delta^{(3)}(x-y). \tag{38}$$

Notice that the Pauli matrices satisfies the commutation relations $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$.

The Hamiltonian operator for the spinor doublet part is now given by

$$H = i\int \mathrm{d}^3x\, \Psi^\dagger \partial_t \Psi. \tag{39}$$

We can compute all commutators in terms of the fundamental anticommutator Eq (38) using repeatedly the manipulation of Eq. (22). We immediately get

$$[Q_a, H] = 0; \qquad [Q, H]. \tag{40}$$

The commutation relations between charges Q_a however are now more interesting. Using the definition of the charge in terms of the Noether current $Q_a = \int \mathrm{d}^3x \mathcal{J}_a^0$, we have:

$$[Q_a, Q_b] = (\sigma_a)_{ij}\,(\sigma_b)_{kl} \int \mathrm{d}^3x\, \mathrm{d}^3y \left[\Psi_i^\dagger(x)\Psi_j(x), \Psi_j^\dagger(y)\Psi_k(y)\right] \tag{41}$$

$$= \int \mathrm{d}^3x\, \mathrm{d}^3y\, \Psi^\dagger(x)\,[\sigma_a, \sigma_b]\,\Psi(x) \tag{42}$$

$$= 2i\varepsilon_{abc} Q_c, \tag{43}$$

where in the first line i and j are the indices in the Lie algebra representation space which allowed us to compute independently the commutation between the spinor doublets. In order to go from the first to the second line, we first expressed the commutator in terms of anticommutators in order to use the relation $\left\{\Psi_i^\dagger(x), \Psi_j(y)\right\} = i\delta_{ij}\delta^{(3)}(x-y)$, then restoring back the commutation between the Pauli matrices. Finally, using the commutation relations between Pauli matrices, it is straightforward to arrive to the final result. Similarly, because the identity matrix of course commutes with Pauli matrices, it is easy to check that

$$[Q, Q_a] = 0. \tag{44}$$

Hence we now find that the maximal numer of operators that commute with the Hamiltonian is given by the charges Q and Q_a, which satisfy the commutation relatons of the generators of the group $[U(1)] \otimes [SU(2)]$.