Solution of the exam of Theoretical Physics of June 28 2023

Real scalar field:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ipx} + a_p^\dagger e^{ipx} \right). \tag{1}$$

Spinor field:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1}^{2} \left(a_p^s u^s(p) e^{-ipx} + b_p^{s\dagger} v^s(p) e^{ipx} \right). \tag{2}$$

1. The energy-momentum tensor is

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial_\nu \psi - \delta^\mu{}_\nu \mathcal{L}. \tag{3}$$

Using the relations

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar{\psi}\gamma^\mu, \tag{4}$$

we find that

$$T^\mu{}_\nu = \partial^\mu \phi \partial_\nu \phi + i\bar{\psi}\gamma^\mu \partial_\nu \psi - \delta^\mu{}_\nu \mathcal{L}. \tag{5}$$

The hamiltonian density is

$$\begin{aligned}
\mathcal{H} = T^{00} &= \partial^0\phi\partial^0\phi + i\psi^\dagger \partial^0 \psi - \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_\phi^2\phi^2\right) - \bar{\psi}\left(i\partial\!\!\!/ - m + ig\phi\right)\psi \\
&= \dot{\phi}^2 + i\psi^\dagger\dot{\psi} - \frac{1}{2}\left(\dot{\phi}^2 + \partial_i\phi\partial^i\phi - m_\phi^2\phi^2\right) - \bar{\psi}\left(i\gamma^0\partial_0 + i\gamma^i\partial_i - m + ig\phi\right)\psi \\
&= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi + \frac{m_\phi^2}{2}\phi^2 + i\bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi - ig\phi\bar{\psi}\psi.
\end{aligned} \tag{6}$$

2. The only internal symmetry of the theory is

$$\psi \to \psi' = e^{-i\theta}\psi, \tag{7}$$

$$\bar{\psi} \to \bar{\psi}' = e^{i\theta}\bar{\psi}. \tag{8}$$

The Noether current is

$$\mathcal{J}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \bar{\psi}\gamma^\mu\psi, \tag{9}$$

where we made use of the expression

$$\Delta\psi = \frac{\delta\psi}{\theta} = \frac{-i\theta\psi}{\theta} = -i\psi. \tag{10}$$

The classical conserved charge is

$$Q_{U(1)} = \int d^3x\, \mathcal{J}^0(x) = \int d^3x\, \bar{\psi}(x)\gamma^0\psi(x) = \int d^3x\, \psi^\dagger(x)\psi(x). \tag{11}$$

In order to write the quantum charge in terms of creation and annihilation operators we substitute Eq. (2) in Eq. (11). We get

$$Q_{U(1)} = \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^3\sqrt{2E_p 2E_{p'}}} \sum_{ss'} \left(a_p^{s\dagger} u^{s\dagger}(p) e^{ipx} + b_p^s v^{s\dagger}(p) e^{-ipx} \right) \left(a_{p'}^{s'} u^{s'}(p') e^{-ip'x} + b_{p'}^{s'\dagger} v^{s'}(p') e^{ip'x} \right) \tag{12}$$

$$= \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^3\sqrt{2E_p 2E_{p'}}} \sum_{ss'} \Big(a_p^{s\dagger} a_{p'}^{s'} u^{s\dagger}(p) u^{s'}(p') e^{i(p-p')x} + a_p^{s\dagger} b_{p'}^{s'\dagger} u^{s\dagger}(p) v^{s'}(p') e^{i(p+p')x} \tag{13}$$

$$+ b_p^s a_{p'}^{s'} v^{s\dagger}(p) u^{s'}(p') e^{-i(p+p')x} + b_p^s b_{p'}^{s'\dagger} v^{s\dagger}(p) v^{s'}(p') e^{-i(p-p')x} \Big). \tag{14}$$

Integrating with respect d^3x we get a delta that removes one of the integrals over momentum and we find

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_{ss'} \Big(a_p^{s\dagger} a_p^{s'} u^{s\dagger}(p) u^{s'}(p) + a_p^{s\dagger} b_{-p}^{s'\dagger} u^{s\dagger}(p) v^{s'}(-p) + b_p^s a_{-p}^{s'} v^{s\dagger}(p) u^{s'}(-p) + b_p^s b_p^{s'\dagger} v^{s\dagger}(p) v^{s'}(p) \Big). \tag{15}$$

Using the normalization conditions

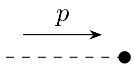
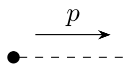
$$u^{s\dagger}(p) u^{s'}(p) = 2E_p \delta^{ss'} \tag{16}$$
$$v^{s\dagger}(p) v^{s'}(p) = 2E_p \delta^{ss'} \tag{17}$$
$$u^{s\dagger}(p) v^{s'}(-p) = 0 \tag{18}$$
$$v^{s\dagger}(p) u^{s'}(-p) = 0 \tag{19}$$

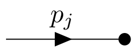
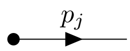
we find

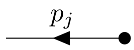
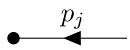
$$Q_{U(1)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1}^{2} \left(a_p^{s\dagger} a_p^s - b_p^{s\dagger} b_p^s \right), \tag{20}$$

where we anticommuted $b_p^{s\dagger}$ and b_p^s and we have removed an infinite additive constant.

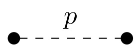
3. • **External lines**

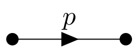
$\phi \, |s(p)\rangle =$  $= 1,$ $\qquad \langle s(p)| \, \phi =$  $= 1,$ (21)

$\psi \, |f(p_j, s)\rangle =$  $= u^s(p_j),$ $\qquad \langle f(p_j, s)| \, \overline{\psi} =$  $= \bar{u}^s(p_j),$ (22)

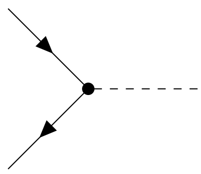
$\overline{\psi} \, |\bar{f}(p_j, s)\rangle =$  $= \bar{v}^s(p_j),$ $\qquad \langle f(p_j, s)| \, \overline{\psi} =$  $= v^s(p_j),$ (23)

• **Propagators**

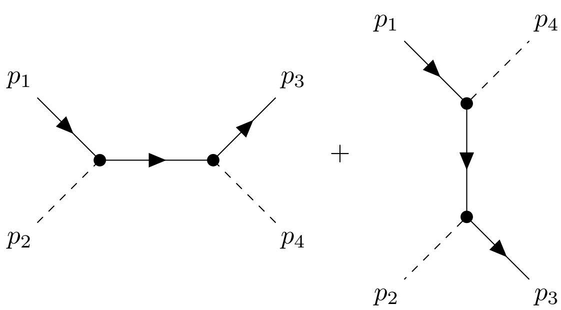
 $= \dfrac{i}{p^2 - m_\phi^2 + i\epsilon},$ scalar propagator (24)

 $= \dfrac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon},$ fermion propagator (25)

• **Vertex**

 $= \; -g,$ (26)

4. The diagrams that contribute to the process $f\phi \to f\phi$ at leading order are



$= i\mathcal{M}_s + i\mathcal{M}_u,$ (27)

The amplitudes corresponding to the twi diagrams are

$$i\mathcal{M}_s = \overline{u}(p_3)(-g) \frac{i(\not{q} + m)}{q^2 - m^2 + i\epsilon} (-g) u(p_1), \quad q = p_1 + p_2 = p_3 + p_4 \tag{28}$$

$$i\mathcal{M}_u = \overline{u}(p_3)(-g) \frac{i(\not{l} + m)}{l^2 - m^2 + i\epsilon} (-g) u(p_1), \quad l = p_1 - p_4 = p_3 - p_2 \tag{29}$$

Note that they contribute with the same sign, because no interchange of fermion fields is necessary. An explicit check is given here (not requested): Using Wick's theorem we get

$$
\begin{aligned}
\langle p_3p_4|iT|p_1p_2\rangle &= \langle p_3p_4|\frac{(-g)^2}{2!}\int d^4x d^4y\, T\Big[\overline{\psi}(x)\psi(x)\phi(x)\overline{\psi}(y)\psi(y)\phi(y)\Big]|p_1p_2\rangle && (30)\\
&= \frac{(-g)^2}{2!}\int d^4xd^4y\,\langle p_3p_4|:\bar{\psi}(x)\psi(x)\phi(x)\bar{\psi}(y)\psi(y)\phi(y):|p_1p_2\rangle && (31)\\
&+ \frac{(-g)^2}{2!}\int d^4xd^4y\,\langle p_3p_4|:\bar{\psi}(x)\psi(x)\phi(x)\bar{\psi}(y)\psi(y)\phi(y):|p_1p_2\rangle && (32)\\
&= \frac{(-g)^2}{2!}\int d^4xd^4y\, S_F(x-y)\langle p_3p_4|:\bar{\psi}(x)\phi(x)\psi(y)\phi(y):|p_1p_2\rangle && (33)\\
&+ \frac{(-g)^2}{2!}\int d^4xd^4y\, S_F(y-x)\langle p_3p_4|:\bar{\psi}(y)\phi(y)\psi(x)\phi(x):|p_1p_2\rangle && (34)\\
&= (-g)^2\int d^4xd^4y\, S_F(x-y)\langle p_3p_4|:\bar{\psi}(x)\phi(x)\psi(y)\phi(y):|p_1p_2\rangle && (35)
\end{aligned}
$$

Now we have to contract the remaining fields with the external particles. All the possible contractions are

$$
\langle p_3p_4|:\bar{\psi}(x)\phi(x)\psi(y)\phi(y):|p_1p_2\rangle + \langle p_3p_4|:\bar{\psi}(x)\phi(x)\psi(y)\phi(y):|p_1p_2\rangle \sim i\mathcal{M}_s + i\mathcal{M}_u. \tag{36}
$$

Therefore, in the second diagram we have to exchange two scalar fields inside the normal ordering, so the two diagrams have the same sign as stated.

5. The modulus squared of the unpolarized amplitude in the $m \to 0$ limit is

$$
\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \frac{1}{2}\sum_{s_1,s_3=1}^{2}\Big(|\mathcal{M}_s|^2 + |\mathcal{M}_u|^2 + \mathcal{M}_s^*\mathcal{M}_u + \mathcal{M}_s\mathcal{M}_u^*\Big)\\
&= \frac{g^4}{2q^4}\mathrm{Tr}\left[\not{p}_3\not{q}\not{p}_1\not{q}\right] + \frac{g^4}{2l^4}\mathrm{Tr}\left[\not{p}_3\not{l}\not{p}_1\not{l}\right] + \frac{g^4}{2q^2l^2}\left\{\mathrm{Tr}\left[\not{p}_1\not{q}\not{p}_3\not{l}\right] + \mathrm{Tr}\left[\not{p}_3\not{q}\not{p}_1\not{l}\right]\right\}\\
&= 2\frac{g^4}{q^4}\left[2(p_1\cdot p_2)(p_2\cdot p_3) - m_\phi^2(p_1\cdot p_3)\right] + 2\frac{g^4}{l^4}\left[2(p_1\cdot p_4)(p_3\cdot p_4) - m_\phi^2(p_1\cdot p_3)\right]\\
&+ 4\frac{g^4}{q^2l^2}\Big[-(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4) - (p_1\cdot p_2)(p_3\cdot p_4)\Big].
\end{aligned} \tag{37}
$$

In the first line we have summed over the spins of the final particles (i.e. the final fermion) and we have averaged over the spins of the initial particles (i.e. the initial fermion). In order to get Eq. (37) we used

$$
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \tag{38}
$$

6. Defining Mandelstam variables as (in the $m \to 0$ limit)

$$
\begin{aligned}
s &= q^2 = (p_1+p_2)^2 = (p_3+p_4)^2 = m_\phi^2 + 2(p_1\cdot p_2) = m_\phi^2 + 2(p_3\cdot p_4) && (39)\\
t &= (p_1-p_3)^2 = (p_4-p_2)^2 = -2(p_1\cdot p_3) = 2m_\phi^2 - 2(p_2\cdot p_4) && (40)\\
u &= l^2 = (p_1-p_4)^2 = (p_3-p_2)^2 = m_\phi^2 - 2(p_1\cdot p_4) = m_\phi^2 - 2(p_2\cdot p_3) && (41)
\end{aligned}
$$

Eq. (37) becomes

$$
|\bar{\mathcal{M}}|^2 = g^4\left(-\frac{su-m_\phi^4}{s^2} - \frac{su-m_\phi^4}{u^2} + \frac{2su-2m_\phi^4}{su}\right) = g^4\left(-\frac{u}{s} + \frac{m_\phi^4}{s^2} + \frac{m_\phi^4}{u^2} - \frac{s}{u} + 2 - \frac{2m_\phi^4}{su}\right) \tag{42}
$$

In the last step we used that $s+u+t = 2m^2 + 2m_\phi^2 \to 2m_\phi^2$ to eliminate the dependence on t.

7. In this case the Feynman rule for the vertex becomes $-g\gamma^5$ and therefore the amplitudes in Eqs. (28-29) become

$$
i\mathcal{M}_s = \overline{u}(p_3)(-g\gamma^5)\frac{i(\not{q}+m)}{q^2-m^2+i\epsilon}(-g\gamma^5)u(p_1), \quad q = p_1+p_2 = p_3+p_4, \tag{43}
$$

$$
i\mathcal{M}_u = \overline{u}(p_3)(-g\gamma^5)\frac{i(\not{l}+m)}{l^2-m^2+i\epsilon}(-g\gamma^5)u(p_1), \quad l = p_1-p_4 = p_3-p_2. \tag{44}
$$

This means that first trace in Eq. (37) is unchanged:

$$\mathrm{Tr}\left[\not{p}_3\gamma^5\not{q}\gamma^5\not{p}_1\gamma^5\not{q}\gamma^5\right] = \mathrm{Tr}\left[\not{p}_3\not{q}\not{p}_1\not{q}\right] \tag{45}$$

where we used the identities $(\gamma^5)^2 = 1$ and that $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$. Hence the results of Eqs. (37,42) don't change. We can understand this by noting that the interaction term in the two Lagrangians Eq. (1) and (2) of the assignment, in terms of chiral components $\psi_R = \frac{1+\gamma_5}{2}\psi$, $\psi_L = \frac{1-\gamma_5}{2}\psi$, have respectively the form Lagrangian has the form

$$ig\bar{\psi}\psi\phi = ig\left(\bar{\psi}_L\psi_R\phi + \bar{\psi}_R\psi_L\phi\right), \tag{46}$$
$$ig\bar{\psi}\gamma_5\psi\phi = ig\left(\bar{\psi}_L\psi_R\phi - \bar{\psi}_R\psi_L\phi\right). \tag{47}$$

But for a massless theory the right and left components propagate independently, so both the t-channel and s-channel amplitudes can be written as the sum of two contributions, with incominf left and propagating right or incoming right and propagating left fermion. The two chiral amplitudes have manifestly the same relative sign both for the scalar and the pseudoscalar interaction so the total amplitude is unchanged.

8. In order to compute the modulus squared of the amplituthe in the limit $m \to \infty$ we have to repeat the computation of the exercise 5, keeping the mass of the fermion. We get

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \frac{1}{2}\sum_{s_1,s_3=1}^{2}\left(|\mathcal{M}_s|^2 + |\mathcal{M}_u|^2 + \mathcal{M}_s^*\mathcal{M}_u + \mathcal{M}_s\mathcal{M}_u^*\right)\\
&= \frac{g^4}{2(q^2-m^2)^2}\mathrm{Tr}\left[(\not{p}_3+m)(\not{q}+m)(\not{p}_1+m)(\not{q}+m)\right] + \frac{g^4}{2(l^2-m^2)^2}\mathrm{Tr}\left[(\not{p}_3+m)(\not{l}+m)(\not{p}_1+m)(\not{l}+m)\right]\\
&+ \frac{g^4}{2(q^2-m^2)(l^2-m^2)}\left\{\mathrm{Tr}\left[(\not{p}_1+m)(\not{q}+m)(\not{p}_3+m)(\not{l}+m)\right] + \mathrm{Tr}\left[(\not{p}_3+m)(\not{q}+m)(\not{p}_1+m)(\not{l}+m)\right]\right\}\\
&= 2\frac{g^4}{(q^2-m^2)^2}\left[4m^2(p_1\cdot p_2) + 4m^2(p_1\cdot p_3) + 4m^2(p_2\cdot p_3) - m_\phi^2(p_1\cdot p_3) + 2(p_1\cdot p_2)(p_2\cdot p_3) + 4m^4 + m^2m_\phi^2\right]\\
&+ 2\frac{g^4}{(l^2-m^2)^2}\left[4m^2(p_1\cdot p_3) - 4m^2(p_1\cdot p_4) - 4m^2(p_3\cdot p_4) - m_\phi^2(p_1\cdot p_3) + 2(p_1\cdot p_4)(p_3\cdot p_4) + 4m^4 + m^2m_\phi^2\right]\\
&+ 4\frac{g^4}{(q^2-m^2)(l^2-m^2)}\Big[2m^2(p_1\cdot p_2) + 4m^2(p_1\cdot p_3) - 2m^2(p_1\cdot p_4) + 2m^2(p_2\cdot p_3) - m^2(p_2\cdot p_4) - 2m^2(p_3\cdot p_4)\\
&- (p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4) - (p_1\cdot p_2)(p_3\cdot p_4) + 4m^4\Big]
\end{aligned} \tag{48}$$

The Mandelstam variables in Eqs. (39-41) with full mass dependence are

$$s = q^2 = (p_1+p_2)^2 = (p_3+p_4)^2 = m^2 + m_\phi^2 + 2(p_1\cdot p_2) = m^2 + m_\phi^2 + 2(p_3\cdot p_4) \tag{49}$$
$$t = (p_1-p_3)^2 = (p_4-p_2)^2 = 2m^2 - 2(p_1\cdot p_3) = 2m_\phi^2 - 2(p_2\cdot p_4) \tag{50}$$
$$u = l^2 = (p_1-p_4)^2 = (p_3-p_2)^2 = m^2 + m_\phi^2 - 2(p_1\cdot p_4) = m^2 + m_\phi^2 - 2(p_2\cdot p_3). \tag{51}$$

Equation (48) becomes

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 = g^4\Bigg(&\frac{15m^4 - 2m^2m_\phi^2 + 5m^2s - 4m^2t - 3m^2u - m_\phi^4 + m_\phi^2s + m_\phi^2t + m_\phi^2u - su}{(s-m^2)^2}\\
&+ \frac{15m^4 - 2m^2m_\phi^2 - 3m^2s - 4m^2t + 5m^2u - m_\phi^4 + m_\phi^2s + m_\phi^2t + m_\phi^2u - su}{(u-m^2)^2}\\
&+ \frac{30m^4 - 4m^2m_\phi^2 + 2m^2s - 8m^2t + 2m^2u - 2m_\phi^4 + 2m_\phi^2s - 2m_\phi^2t + 2m_\phi^2u - s^2 + t^2 - u^2}{(s-m^2)(u-m^2)}\Bigg).
\end{aligned} \tag{52}$$

Taking now the limit $m \to \infty$ the Mandelstam invariants are

$$s = (p_1+p_2)^2 = m^2 + m_\phi^2 + 2p_1\cdot p_2 = m^2 + m_\phi^2 + 2\left(\sqrt{m^2+|\vec{p}_1|^2}\sqrt{m_\phi^2+|\vec{p}_2|^2} - \vec{p}_1\cdot\vec{p}_2\right) \to m^2 + 2mm_\phi \tag{53}$$

and

$$u = (p_1-p_4)^2 = m^2 + m_\phi^2 - 2p_1\cdot p_4 = m^2 + m_\phi^2 - 2\left(\sqrt{m^2+|\vec{p}_1|^2}\sqrt{m_\phi^2+|\vec{p}_4|^2} - \vec{p}_1\cdot\vec{p}_4\right) \to m^2 - 2mm_\phi \tag{54}$$

and we find

$$|\bar{\mathcal{M}}|^2 \to g^4\left(32 + 8\frac{m_\phi^2}{m^2}\right) \to 32g^4. \tag{55}$$

Because the coupling g is dimensionless, in the $m \to \infty$ limit in which all other dimensionful quantities are negligible the squared amplitude can only depend on g by dimensional analysis. Note that in the Mandelstam invariants we must expand up to first order in $1/m$ because in the propagator denominators the leading order term cancels. Note also that, because the result cannot depend on the momenta, when performing the expansion in Eq. (53) we are free to choose $|\vec{p}_i| << m_\phi$.

On this point full score is given for understanding that the result is constant and providing a correct argument for estimating this constant even in the absence of a full calculation.