Solutions to the exam of QFT1 of 19 June 2025

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - M^2\phi^2) + \bar{\psi}_1(i\partial\!\!\!/ - m)\psi_1 + \bar{\psi}_2(i\partial\!\!\!/ - m)\psi_2 - g(\bar{\psi}_1\psi_2\phi + \bar{\psi}_2\psi_1\phi) \tag{1}$$

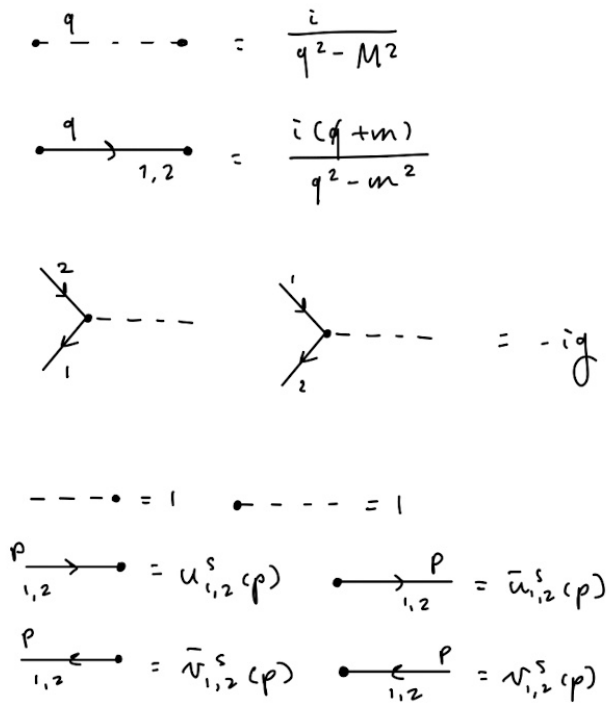
(1) The energy-momentum tensor is

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_1)}\partial^\nu\psi_1 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_2)}\partial^\nu\psi_2 - \eta^{\mu\nu}\mathcal{L} \\ &= \partial^\mu\phi\partial^\nu\phi + i\bar{\psi}_1\gamma^\mu\partial^\nu\psi_1 + i\bar{\psi}_2\gamma^\mu\partial^\nu\psi_2 - \eta^{\mu\nu}\mathcal{L}. \end{aligned} \tag{2}$$

The terms with the barred fermion fields are zero because the Lagrangian is not dependent on their derivative. The Hamiltonian density is the 0'th component of this:

$$H = T^{00} = \frac{1}{2}(\dot{\phi}^2 + (\nabla\phi)^2 + M^2\phi^2) - i\bar{\psi}_1\vec{\gamma}\cdot\nabla\psi_1 + m\bar{\psi}_1\psi_1 - i\bar{\psi}_2\vec{\gamma}\cdot\nabla\psi_2 + m\bar{\psi}_2\psi_2 + g(\bar{\psi}_1\psi_2\phi + \bar{\psi}_2\psi_1\phi). \tag{3}$$

(2) You can read off the Feynman rules from the Lagrangian:



(3) Consider the transformation

$$\begin{aligned}
\psi_1 &\to e^{-i\theta_1}\psi_1 \\
\psi_2 &\to e^{-i\theta_2}\psi_2
\end{aligned} \tag{4}$$

The Lagrangian is invariant under this transformation as long as $\theta_1 = \theta_2 = \theta$. To construct the Noether current we need the corresponding infenitesimal transformations of the fields:

$$\begin{aligned}
\psi_1 &\to (1 - i\theta)\psi_1 \\
\psi_2 &\to (1 - i\theta)\psi_2 \\
\to \delta\psi_1 &= -i\theta\psi_1, \delta\psi_2 = i\theta\psi_2.
\end{aligned} \tag{5}$$

The Noether current is then given by

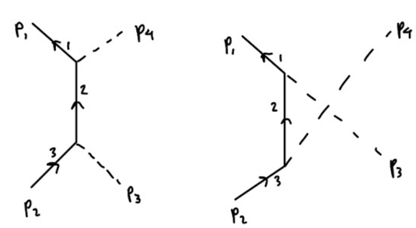
$$\begin{aligned}
J^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_1)}\frac{\delta\psi_1}{\theta} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_2)}\frac{\delta\psi_2}{\theta} \\
&= \bar{\psi_1}\gamma^\mu\psi_1 + \bar{\psi_2}\gamma^\mu\psi_2.
\end{aligned} \tag{6}$$

The corresponding conserved charge is

$$\begin{aligned}
Q &= \int d^3x J^0(x) \\
&= \int d3x\left(\bar{\psi_1}\gamma^0\psi_1 + \bar{\psi_2}\gamma^0\psi_2\right) \\
&= \int d3x\left(\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2\right)
\end{aligned} \tag{7}$$

The physical meaning of is the conservation of the number of fermions.

(4) We can build the interaction $\bar{f}_1 f_1 \to \phi\phi$ from the Feynman rules, and we find two tree-level diagrams: the u-channel and the t-channel.



(5) The total unpolarized amplitude is given by

$$\frac{1}{2}\sum_{s_1,s_2}|M|^2=\frac{1}{2}\sum_{s_1,s_2}|M_u|^2+|M_t|^2+M_u^*M_t+M_t^*M_u, \tag{8}$$

with

$$\begin{aligned}M_u&=-g^2\bar{v}_1(p_1)\frac{i(\not{q}+m)}{q^2-m^2}u_1(p_2),\\ M_t&=-ig^2\bar{v}_1(p_1)\frac{\not{k}+m}{k^2-m^2}u_1(p_2),\\ q&=p_1-p_4,\ k=p_1-p_3.\end{aligned} \tag{9}$$

For the squared amplitudes we need to use the gamma trace identity $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)=4(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho})$, the fact that a trace over an odd number of gamma matrices is zero, and the polarization sums. Defining also $u=(p_1-p_4)^2$ and $t=(p_1-p_3)^2$ and setting $m=0$ we get

$$\begin{aligned}\overline{|M_u|}^2&=\frac{1}{4}\sum_{s_1,s_2}g^4\bar{v}_1(p_1)\frac{\not{p}_1-\not{p}_4}{(p_1-p_4)^2}u_1(p_2)v_1(p_1)\frac{\not{p}_1-\not{p}_4}{(p_1-p_4)^2}\overline{u}_1(p_2)\\ &=\frac{g^4}{u^2}(2(p_1\cdot p_4)(p_2\cdot p_4)-M^2(p_1\cdot p_2));\\ \overline{|M_t|}^2&=\frac{g^4}{t^2}(2(p_1\cdot p_3)(p_2\cdot p_3)-M^2(p_1\cdot p_2));\\ \overline{M_uM_t^*}&=\frac{g^4}{ut}((p_1\cdot p_4)(p_2\cdot p_3)-(p_1\cdot p_2)(p_4\cdot p_3)+(p_1\cdot p_3)(p_4\cdot p_2));\\ \overline{M_u^*M_t}&=\frac{g^4}{ut}((p_1\cdot p_3)(p_2\cdot p_4)-(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_3\cdot p_2)).\end{aligned} \tag{10}$$

In total the spin-averaged amplitude is then

$$\begin{aligned}\overline{|M|^2}=g^4\Big[&\frac{2(p_1\cdot p_4)(p_2\cdot p_4)-M^2(p_1\cdot p_2)}{u^2}\\ &+\frac{2(p_1\cdot p_3)(p_2\cdot p_3)-M^2(p_1\cdot p_2)}{t^2}\\ &+2\frac{(p_1\cdot p_3)(p_2\cdot p_4)-(p_1\cdot p_2)(p_3\cdot p_4+(p_1\cdot p_4)(p_2\cdot p_3)}{ut}\Big].\end{aligned} \tag{11}$$

For completeness we have kept all masses but full score is given for the computation with $m=0$ as requested in the assignment.

(6) Now we set $m=0$ and write everything in terms of Mandelstam variables. The momenta can be written as

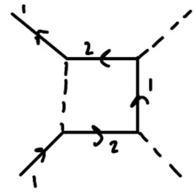


Figure 1: $D = 4 \cdot 1 - 2 \cdot 1 - 3 \cdot 1 = -1$. The box diagram is UV-finite.

$$(p_1 \cdot p_2) = \frac{s}{2}; \tag{12}$$

$$(p_3 \cdot p_4) = \frac{s}{2} - M^2; \tag{13}$$

$$(p_1 \cdot p_4) = (p_2 \cdot p_3) = \frac{M^2 - u}{2}; \tag{14}$$

$$(p_1 \cdot p_3) = (p_2 \cdot p_4) = \frac{M^2 - t}{2}. \tag{15}$$

Another thing that is useful to realize is that $s + t + u = 2M^2$. Some careful rewriting yields

$$\overline{|M|^2} = \frac{g^4}{2}\left[\frac{ut - M^4}{u^2} + \frac{ut - M^4}{t^2} + 2\frac{M^4 - ut}{ut}\right] \tag{16}$$

$$= \frac{g^4}{2}\left(ut - M^4\right)\frac{(u+t)^2}{t^2u^2}. \tag{17}$$

(7) The t-channel one-loop diagrams are shown in Figs. 1-4. All these diagrams have a u–channel counterpart. We have one box diagram, the fermion and scalar self-energy corrections and the vertex correction. We can calculate the degree of divergence by using the formula $D = 4L - 2P_s - P_f$, because a loop gives 4 powers of momenta upstairs from the integral, the scalar propagator 2 powers downstairs, and the fermion propagator one power downstairs. Note that the superficial degree of divergence of the self-energy and vertex diagrams is actually lower than the actual degree of divergence because they are divergent subdiagrams of a diagram with more propagators.

(8) Writing $\Psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$, it follows that $\bar{\Psi} = (\bar{\psi}_1, \bar{\psi}_2)$ so the interaction Lagrangian can be written as

$$\mathcal{L}_{int} = \bar{\Psi}\begin{pmatrix}0 & -g\phi \\ -g\phi & 0\end{pmatrix}\Psi = -g\phi\bar{\Psi}\sigma_1\Psi, \tag{18}$$

where σ_1 is the first Pauli matrix. It is then clear that the Lagrangian is also invariant under the transformation $\Psi \to e^{i\alpha\sigma_1}\Psi$.

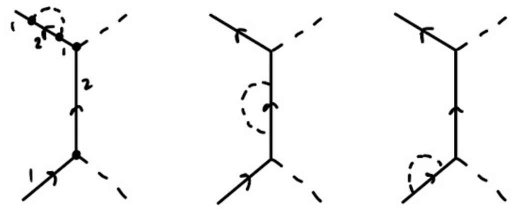


Figure 2: $D = 4 \cdot 1 - 2 \cdot 1 - 1 = 1$. The fermion self-energy diagrams are linearly divergent.

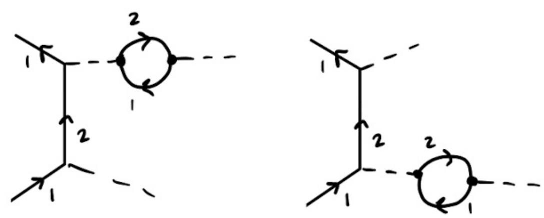


Figure 3: $D = 4 \cdot 1 - 2 = 1$. The scalar self-energy diagrams are quadratically divergent.

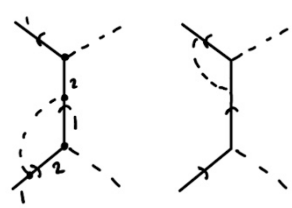


Figure 4: $D = 4 \cdot 1 - 2 \cdot 1 - 2 = 0$. The vertex correction is logarithmically divergent.