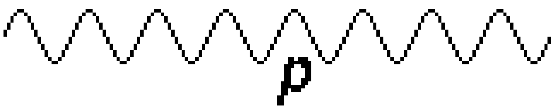
Quantum Field Theory I: written test solution

July 19, 2017

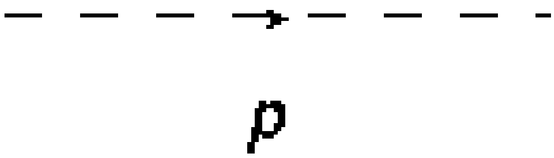
1. From the Lagrangian

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) + \frac{g}{4}\phi F_{\mu\nu}F^{\mu\nu}. \tag{1}$$

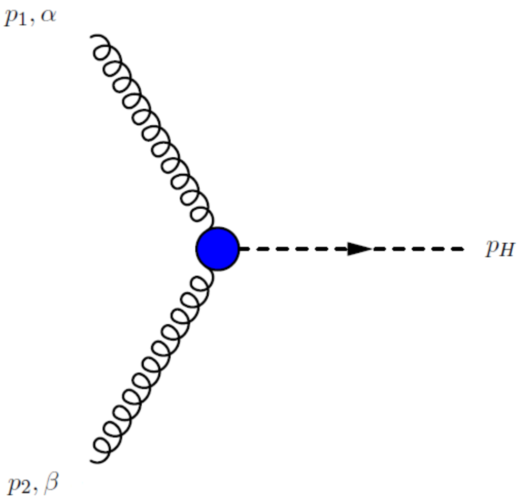
we get the Feynman rules



$$\frac{-ig^{\alpha\beta}}{p^2 + i\epsilon} \tag{2}$$



$$\frac{i}{p^2 - m^2 + i\epsilon} \tag{3}$$



$$-\,ig\left(\left(p_1 \cdot p_2\right)g^{\alpha\beta} - p_1^\beta p_2^\alpha\right) \tag{4}$$

where incoming momenta flow towards the vertex.
The dimension of g is $[M]^{-1}$, the inverse of a mass.

2. For Higgs production at lowest order there is only one diagram, which coincides with the vertex. The matrix element is thus

$$i\mathcal{M} = -ig\,\epsilon_\alpha\left(p_1\right)\epsilon_\beta\left(p_2\right)\left(\left(p_1 \cdot p_2\right)g^{\alpha\beta} - p_1^\beta p_2^\alpha\right), \tag{5}$$

where we have recalled the fact that the Feynman rules provide an expression for the S-matrix element $i\mathcal{M}$.

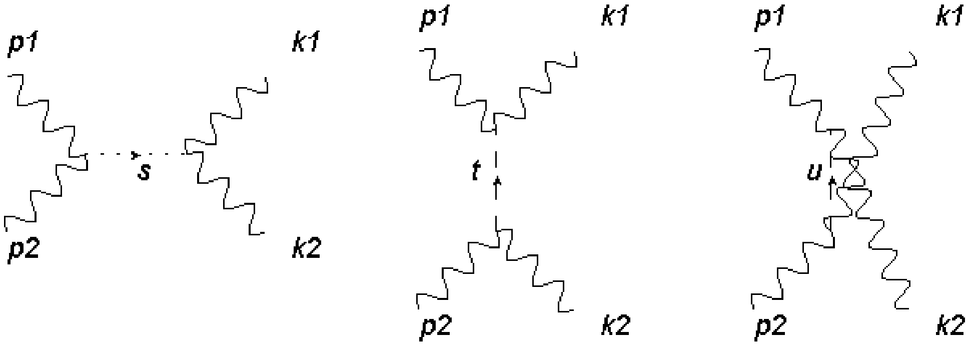


Figure 1: LO diagrams to $\gamma+\gamma\to\gamma+\gamma$

The unpolarized squared amplitude is

$$\begin{aligned}\frac{1}{4}\sum|\mathcal{M}|^2 &= \frac{g^2}{4}\left(-g_{\alpha\mu}\right)\left(-g_{\beta\nu}\right)\left(\left(p_1\cdot p_2\right)g^{\alpha\beta}-p_1^{\beta}p_2^{\alpha}\right)\left(\left(p_1\cdot p_2\right)g^{\mu\nu}-p_1^{\nu}p_2^{\mu}\right)\\ &= \frac{g^2}{4}\left(\left(p_1\cdot p_2\right)g^{\alpha\beta}-p_1^{\beta}p_2^{\alpha}\right)\left(\left(p_1\cdot p_2\right)g_{\alpha\beta}-p_{1\beta}p_{2\alpha}\right)\\ &= \frac{g^2}{4}\left(4\left(p_1\cdot p_2\right)^2-\left(p_1\cdot p_2\right)^2-\left(p_1\cdot p_2\right)\right)=\frac{g^2}{2}\left(p_1\cdot p_2\right)^2 \end{aligned}\tag{6}$$

where we have used the polarizarion sum given in the assignment

$$\sum\epsilon_{\alpha}\left(p\right)\epsilon_{\mu}^{*}\left(p\right)=-g_{\alpha\mu}. \tag{7}$$

In terms of Mandelstam variables, Eq. (6) becomes

$$\frac{1}{4}\sum|\mathcal{M}|^2=\frac{g^2}{8}s^2, \tag{8}$$

given that $s=\left(p_1+p_2\right)^2=2\left(p_1\cdot p_2\right)$.

3. The process

$$\gamma\left(p_1\right)+\gamma\left(p_2\right)\to\gamma\left(k_1\right)+\gamma\left(k_2\right) \tag{9}$$

proceeds through the three Feynman diagrams shown in Fig. 3, corresponding to s-, t-, and u-channel contributions.

The corresponding amplitudes are

$$i\mathcal{M}_s=-i\frac{g^2}{s-m^2+i\epsilon}\epsilon_{\alpha}\left(p_1\right)\epsilon_{\beta}\left(p_2\right)\epsilon_{\mu}^{*}\left(k_1\right)\epsilon_{\nu}^{*}\left(k_2\right)\left(\left(p_1\cdot p_2\right)g^{\alpha\beta}-p_1^{\beta}p_2^{\alpha}\right)\left(\left(k_1\cdot k_2\right)g^{\mu\nu}-k_1^{\nu}k_2^{\mu}\right) \tag{10}$$

$$i\mathcal{M}_t=-i\frac{g^2}{t-m^2+i\epsilon}\epsilon_{\alpha}\left(p_1\right)\epsilon_{\beta}\left(p_2\right)\epsilon_{\mu}^{*}\left(k_1\right)\epsilon_{\nu}^{*}\left(k_2\right)\left(-\left(p_1\cdot k_1\right)g^{\alpha\mu}+p_1^{\mu}k_1^{\alpha}\right)\left(-\left(p_2\cdot k_2\right)g^{\beta\nu}+p_{\nu}k_2^{\beta}\right) \tag{11}$$

$$i\mathcal{M}_u=-i\frac{g^2}{u-m^2+i\epsilon}\epsilon_{\alpha}\left(p_1\right)\epsilon_{\beta}\left(p_2\right)\epsilon_{\mu}^{*}\left(k_1\right)\epsilon_{\nu}^{*}\left(k_2\right)\left(-\left(p_1\cdot k_2\right)g^{\alpha\nu}+p_1^{\nu}k_2^{\alpha}\right)\left(-\left(k_1\cdot p_2\right)g^{\mu\beta}+k_1^{\beta}p_2^{\mu}\right). \tag{12}$$

The total amplitude for this process is given by the sum of the three contributions:

$$\mathcal{M}=\mathcal{M}_s+\mathcal{M}_t+\mathcal{M}_u. \tag{13}$$

4. Keeping only the s-channel contribution $\mathcal{M}_s$ we get

$$\begin{aligned}\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{1}{4}\sum|\mathcal{M}_s|^2 &= \frac{g^4}{4\left(s-m^2\right)^2}\left(\frac{s}{2}g^{\alpha\beta}-p_1^\beta p_2^\alpha\right)\left(\frac{s}{2}g^{\mu\nu}-k_1^\nu k_2^\mu\right)\\ &\quad\left(\frac{s}{2}g_{\alpha\beta}-p_{1\beta}p_{2\alpha}\right)\left(\frac{s}{2}g_{\mu\nu}-k_{1\nu}k_{2\mu}\right)\\ &= \frac{g^4}{4\left(s-m^2\right)^2}\left(s^2-\frac{s^2}{2}\right)\left(s^2-\frac{s^2}{2}\right) = \frac{g^4 s^4}{16\left(s-m^2\right)^2},\end{aligned} \tag{14}$$

where we have set $\epsilon = 0$ in the propagators, and we have made use of

$$(p_1\cdot p_2) = (k_1\cdot k_2) = \frac{s}{2}. \tag{15}$$

5. For Higgs production the phase space is one-body:

$$\begin{aligned}d\Phi_1 &= \frac{d^3 p_H}{(2\pi)^3\, 2m}(2\pi)^4\,\delta^{(4)}\left(p_1+p_2-p_H\right)\\ &= \frac{\pi}{m}\delta\left(\sqrt{s}-m\right) = 2\pi\delta\left(s-m^2\right)\end{aligned} \tag{16}$$

and the flux factor is

$$\Phi = 4\sqrt{(p_1\cdot p_2)^2} = 2s \tag{17}$$

The total cross section is therefore

$$\sigma_1 = \frac{g^2 m^2}{8}\pi\delta\left(s-m^2\right). \tag{18}$$

6. For photon-photon scattering the phase space is the standard two-body

$$d\Phi_2 = \frac{1}{2}\frac{d^3k_1}{(2\pi)^3\sqrt{s}}\frac{d^3k_2}{(2\pi)^3\sqrt{s}}(2\pi)^4\,\delta^{(4)}\left(p_1+p_2-k_1-k_2\right) \tag{19}$$

since in the center-of-mass frame, with all particle massless, we have

$$|p_1|^2 = |p_2|^2 = |k_1|^2 = |k_2|^2 = k = \frac{\sqrt{s}}{2}, \tag{20}$$

and we have to halve the result since in the final state we have two indistinguishable particles.

We can simplify the phase space as

$$\begin{aligned}d\Phi_2 &= \frac{1}{2}\frac{dk\,d\cos\theta\,d\phi}{16\pi^2}\delta\left(\sqrt{s}-2k\right)\\ &= \frac{d\cos\theta}{32\pi}\end{aligned} \tag{21}$$

and we can evaluate the flux factor as before

$$\Phi = 2s. \tag{22}$$

We thus obtain the following implicit expression for the differential cross section:

$$\frac{d\sigma_2}{d\cos\theta} = \frac{1}{64\pi s}\frac{1}{4}\sum|\mathcal{M}|^2 \tag{23}$$

with $\mathcal{M}$ given by Eq. (13).

7. We want to show that

$$\mathrm{Im} M_2 = \frac{\Phi}{2}\sigma_1. \tag{24}$$

We start from Eq. (10), set $k_1 = p_1$, $k_2 = p_2$, and perform the average over initial polarizations: we get

$$\frac{1}{4}\sum \mathcal{M}_s\left(p_1 + p_2 \to p_1 + p_2\right) = -\frac{g^2}{8\left(s - m^2 + i\epsilon\right)}s^2, \tag{25}$$

where we have restored the crucial factor of $i\epsilon$ in the propagator. Using the hint

$$\lim_{\epsilon\to 0}\frac{1}{s - m^2 + i\epsilon} = \frac{1}{s - m^2} - i\pi\delta\left(s - m^2\right), \tag{26}$$

so we can rewrite Eq. (25) as

$$\frac{1}{4}\sum \mathcal{M}_s\left(p_1 + p_2 \to p_1 + p_2\right) = -\frac{g^2 s^2}{8\left(s - m^2\right)} + i\pi\frac{g^2 m^2}{8}\delta\left(s - m^2\right). \tag{27}$$

We thus get immediately

$$\mathrm{Im}\left(\frac{1}{4}\sum \mathcal{M}_s\left(p_1 + p_2 \to p_1 + p_2\right)\right) = \pi\frac{g^2 m^2}{8}\delta\left(s - m^2\right) = m^2\sigma_1. \tag{28}$$

Comparing the Feynman diagram of Fig 3 and the vertex Eq. (4) it is clear that the former can be viewed as the “square modulus” of the latter, when the propagator goes on shell. The identity Eq. (26) then extracts this on-shell part.