QUANTUM FIELD THEORY I
Solution
July 23, 2019

Consider the scalar QED Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^* (D^\mu\phi) - m^2\phi^*\phi, \tag{1}$$

where ϕ is a complex scalar field, $F_{\mu\nu}$ is the Maxwell field strength, and D_μ is the covariant derivative defined as

$$D_\mu = \partial_\mu + ieA_\mu. \tag{2}$$

- The complex scalar field ϕ is defined as

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left\{ a_{\vec{p}}\, \mathrm{e}^{-ip^\mu x_\mu} + b^\dagger_{\vec{p}}\, \mathrm{e}^{ip^\mu x_\mu} \right\}, \tag{3}$$

  with the commutation relation:

$$[a_{\vec{p}}, a^\dagger_{\vec{k}}] = [b_{\vec{p}}, b^\dagger_{\vec{k}}] = (2\pi)^3\delta^{(3)}(\vec{p} - \vec{k}). \tag{4}$$

- The Maxwell field tensor is defined as $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$, where the expression of the photon field is given by

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_\lambda \left\{ \epsilon_{\mu,\lambda}(p) a_{\vec{p},\lambda}\, \mathrm{e}^{-ip^\mu x_\mu} + \epsilon^*_{\mu,\lambda}(p) a^\dagger_{\vec{p}}\, \mathrm{e}^{ip^\mu x_\mu} \right\}. \tag{5}$$

(1) The energy-momentum tensor of this theory is given by:

$$T^\mu_{\ \nu} = \frac{\partial\mathcal{L}'}{\partial(\partial_\mu\phi)}\partial_\nu\phi + \frac{\partial\mathcal{L}'}{\partial(\partial_\mu\phi^*)}\partial_\nu\phi^* + \frac{\partial\mathcal{L}'}{\partial(\partial_\mu A_\lambda)}\partial_\nu A_\lambda - \delta^\mu_{\ \nu}\mathcal{L}', \tag{6}$$

where $\mathcal{L}' = |D\phi|^2 - m^2|\phi|^2$. Hence, we have

$$T^\mu_{\ \nu} = (\partial^\mu - ieA^\mu)\phi^*\partial_\nu\phi + (\partial^\mu + ieA^\mu)\phi\partial_\nu\phi^* - \delta^\mu_{\ \nu}\mathcal{L}'. \tag{7}$$

The Hamiltonian density $\mathcal{H} = T^{00}$ is then given by

$$\mathcal{H} = 2\dot{\phi}\dot{\phi}^* + ieA^0(\phi\dot{\phi}^* - \phi^*\dot{\phi}) - |D\phi|^2 + m^2|\phi|^2 \tag{8}$$

$$= \dot{\phi}\dot{\phi}^* + \vec{\nabla}\phi^*\cdot\vec{\nabla}\phi + ie\left[\phi(\vec{\nabla}\phi^*)\cdot\vec{A} - \phi^*(\vec{\nabla}\phi)\cdot\vec{A}\right] + \left(m^2 - e^2A_\mu A^\mu\right)|\phi|^2, \tag{9}$$

where $\dot{\phi}$ denotes $\partial_0\phi$ (and resp. $\dot{\phi}^*$ denotes $\partial_0\phi^*$).

(2) The Lagrangian is invariant under the transformation

$$\phi \longrightarrow \phi' = \mathrm{e}^{-i\alpha}\phi. \tag{10}$$

Because the field is complex, the variations of the fields ϕ and ϕ^* must be considered as independent. The infinitesimal transformations are

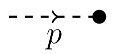
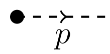
$$\phi \longrightarrow \phi' = (1 - i\alpha)\phi \Rightarrow \Delta\phi = -i\phi \tag{11}$$
$$\phi^* \longrightarrow (\phi')^* = (1 + i\alpha)\phi^* \Rightarrow \Delta\phi^* = i\phi^* \tag{12}$$

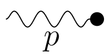
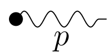
Using these results to compute the conserved Noether current associated to the above symmetry, we have:

$$\mathcal{J}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\Delta\phi^* \tag{13}$$
$$= (\partial^\mu \phi^* - ieA^\mu \phi^*)(-i\phi) + (\partial^\mu \phi + ieA^\mu \phi)(i\phi^*) \tag{14}$$
$$= i(\phi^* D^\mu \phi - \phi D^\mu \phi^*). \tag{15}$$

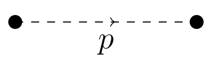
(3) The Feynman rules of the theory are

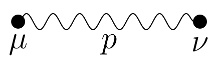
(a) External lines:

$\phi|s(p)\rangle =$  $= 1,$ $\quad$ $\langle s(p)|\phi =$  $= 1,$

$A_\mu|\gamma(p,s)\rangle =$  $= \epsilon_\mu(p,s),$ $\quad$ $\langle \gamma(p,s)|A_\mu =$  $= \epsilon^*_\mu(p,s).$

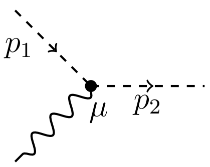
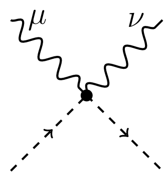
(b) Propagators:

Scalar :  $= \dfrac{i}{p^2 - m^2 + i\epsilon}$ (16)

Photon :  $= -\dfrac{i}{p^2 + i\epsilon}\left[g_{\mu\nu} - (1-\xi)\dfrac{p_\mu p_\nu}{p^2}\right].$ (17)

The Feynman gauge expression $\xi = 1$ is considered as a correct solution.

(c) Vertices:

 $= -ie(p_1 + p_2)^\mu,$  $= 2ie^2 g^{\mu\nu}$ (18)

One can deduce from the Lagrangian that since $[\partial_\mu] = [\phi] = [A_\mu] = 1$ the coupling e is dimensionless and therefore the theory is renormalizable.

(4) As shown in Fig. 1, three diagrams contribute to the process $\gamma\phi \to \gamma\phi$. The respective contribtions to the amplitude are

$$i\mathcal{M}_g(\gamma\phi \to \gamma\phi) = \epsilon^*_\mu(p_4)\left[2ie^2 g^{\mu\nu}\right]\epsilon_\nu(p_2) \tag{19}$$
$$i\mathcal{M}_s(\gamma\phi \to \gamma\phi) = (-ie)^2 \epsilon^*_\mu(p_4)(p_3 + k)^\mu \left(\frac{i}{k^2 - m^2}\right)(p_1 + k)^\nu \epsilon_\nu(p_2) \tag{20}$$
$$i\mathcal{M}_u(\gamma\phi \to \gamma\phi) = (-ie)^2 \epsilon^*_\mu(p_4)(p_1 + l)^\mu \left(\frac{i}{l^2 - m^2}\right)(p_3 + l)^\nu \epsilon_\nu(p_2) \tag{21}$$

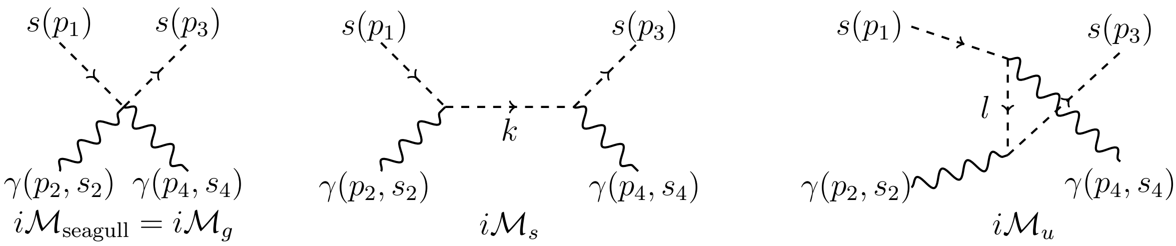


Figure 1: Leading non-vanishing diagrams for $\gamma\phi \to \gamma\phi$.

The scattering amplitude is found as the sum of these contributions. It can be simplified using momentum conservation and the fact that $\epsilon_\mu(p_i)p_i^\mu = 0$. One gets

$$\mathcal{M}(\gamma\phi \to \gamma\phi) = e^2\epsilon^*_\mu(p_4)\epsilon_\nu(p_2)\left(\mathrm{M}_g^{\mu\nu} + \mathrm{M}_s^{\mu\nu} + \mathrm{M}_u^{\mu\nu}\right) = 2e^2\epsilon^*_\mu(p_4)\epsilon_\nu(p_2)\mathrm{M}^{\mu\nu}, \quad (22)$$

where $\mathrm{M}^{\mu\nu}$ is defined as

$$\mathrm{M}^{\mu\nu} = g^{\mu\nu} - \frac{p_3^\mu p_1^\nu}{(p_1p_2)} + \frac{p_1^\mu p_3^\nu}{(p_2p_3)}. \quad (23)$$

In the last expression, the parentheses represent the usual scalar products.

(5) In order to show that the amplitude computed in the previous point (Eq. (22)) vanishes if any of the polarization vectors for the external photon is proportional to the momentum carried out by that respective photon we choose $\epsilon_\nu(p_2) = \lambda(p_2)_\nu$. We get

$$\begin{aligned}
\mathcal{M}(\gamma\phi \to \gamma\phi) &= 2\lambda e^2\epsilon^*_\mu(p_4)(p_2)_\nu\left\{g^{\mu\nu} - \frac{p_3^\mu p_1^\nu}{(p_1p_2)} + \frac{p_1^\mu p_3^\nu}{(p_2p_3)}\right\} && (24)\\
&= 2\lambda e^2\epsilon^*_\mu(p_4)\left(p_2 - p_3 + p_1\right)^\mu && (25)\\
&= 2e^2\epsilon^*_\mu(p_4)p_4^\mu && (26)\\
&= 0, && (27)
\end{aligned}$$

where ee used momentum conservation and the fact that $\epsilon^*_\mu(p_4)p_4^\mu = 0$.

(6) We compute the modulus square of the amplitude computed at point (4). Multiplying Eq. (22) by its complex conjugate yields

$$|\mathcal{M}(\gamma\phi \to \gamma\phi)|^2 = 4e^4\epsilon^*_\mu(p_4)\epsilon_\nu(p_2)\epsilon_\rho(p_4)\epsilon^*_\beta(p_2)\mathrm{M}^{\mu\nu}\mathrm{M}^{\rho\beta}. \quad (28)$$

Since we are considering an unpolarized process, we have to sum over the polarization of the final state and average over the polarization of the initial state photons. Writing explicitly the indices of the polarizations, we have

$$\begin{aligned}
|\overline{\mathcal{M}}(\gamma\phi \to \gamma\phi)|^2 &= \frac{1}{2}\sum_{\text{pol.}}|\mathcal{M}(\gamma\phi \to \gamma\phi)|^2 && (29)\\
&= 2e^4\sum_{\lambda,\sigma}\epsilon^*_{\mu,\lambda}(p_4)\epsilon_\nu(p_2,\sigma)\epsilon_{\rho,\lambda}(p_4)\epsilon^*_{\beta,\sigma}(p_2)\mathrm{M}^{\mu\nu}\mathrm{M}^{\rho\beta}. && (30)
\end{aligned}$$

The sum over polarizations is

$$\sum_{\lambda} \epsilon^{*}_{\mu,\lambda}(p_4)\epsilon_{\rho,\lambda}(p_4) = -g_{\mu\rho}. \tag{31}$$

By defining the following quantities

$$\xi^{\mu\nu} = \frac{p_3^{\mu} p_1{}^{\nu}}{(p_1 p_2)}, \qquad \chi^{\mu\nu} = \frac{p_1^{\mu} p_3^{\nu}}{(p_2 p_3)}, \tag{32}$$

and using the result in Eq. (31), the modulus square of the amplitude is given by

$$|\overline{\mathcal{M}}(\gamma\phi \to \gamma\phi)|^2 = 2e^4 \mathrm{M}^{\mu\nu}\mathrm{M}_{\mu\nu} \tag{33}$$

$$= 2e^4 \left(\xi^{\mu\nu}\xi_{\mu\nu} + \chi^{\mu\nu}\chi_{\mu\nu} - 2\xi^{\mu\nu}\chi_{\mu\nu} - 2\xi^{\mu}_{\mu} + 2\chi^{\mu}_{\mu} + 4\right), \tag{34}$$

where

$$\xi^{\mu\nu}\xi_{\mu\nu} = \frac{m^4}{(p_1 p_2)^2}, \quad \xi^{\mu\nu}\chi_{\mu\nu} = \frac{(p_1 p_3)^2}{(p_1 p_2)(p_2 p_3)}, \quad \text{and} \quad \xi^{\mu}_{\mu} = \frac{(p_1 p_3)}{(p_1 p_2)}. \tag{35}$$

Notice that ξ and χ are related by crossing symmetry. Thus, the expression of χ can be obtained from ξ by replacing p_2 with (-p_4) and vice-versa. With a little bit of algebra, we finally find that the final expression of the amplitude square is given by the following expression

$$|\overline{\mathcal{M}}(\gamma\phi \to \gamma\phi)|^2 = 2e^4 \left\{ \frac{m^4}{(p_1 p_2)^2} + \frac{m^4}{(p_2 p_3)^2} - \frac{2(p_1 p_3)^2}{(p_1 p_2)(p_2 p_3)} \right. \tag{36}$$

$$\left. - \frac{2(p_1 p_3)}{(p_1 p_2)} + \frac{2(p_1 p_3)}{(p_2 p_3)} + 4 \right\}. \tag{37}$$

(7) The canonical commutation relations for the scalar QED are given by

$$[\phi(x), \pi(y)] = [\phi^*(x), \pi^*(y)] = i\delta^{(3)}(x-y), \quad [A_\mu(x), \tilde{\pi}_\nu(y)] = ig_{\mu\nu}\delta^{(3)}(x-y),$$

where the canonical momenta are given by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^* - ieA_0\phi^*, \quad \pi^*(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi} + ieA_0\phi, \tag{38}$$

$$\pi_\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu}, \quad \text{with} \quad \pi_0(x) = -\partial_\mu A^\mu \quad \text{and} \quad \pi_i(x) = \partial_i A_0 - \dot{A}_i. \tag{39}$$

Notice that all the other commutation relations vanish. On the other hand, using the expression of of the Noether current in Eq. (15), the Noether charge can be written in the following form

$$\mathcal{Q} = \int \mathrm{d}^3x \left[i(\phi^*\dot{\phi} - \phi\dot{\phi}^*) - 2eA_0\phi^*\phi \right] \tag{40}$$

$$= i \int \mathrm{d}^3x \, (\phi^*\pi^* - \phi\pi). \tag{41}$$

We can now compute the commutator if the Noether charge with the fields in the scalar QED theory:

$$[\mathcal{Q}, \phi(x)] = -i \int \mathrm{d}^3x\, \phi(x)[\pi(y), \phi(x)] = -\phi(x), \tag{42}$$

$$[\mathcal{Q}, \phi^*(x)] = i \int \mathrm{d}^3x\, \phi^*(x)[\pi(y), \phi^*(x)] = \phi^*(x), \tag{43}$$

$$[\mathcal{Q}, A_\mu(x)] = 0. \tag{44}$$

It follows that at the quantum level the classical Noether charge generates the infinitesimal transformation upon commutation.