QUANTUM FIELD THEORY I

Solution

July 14th, 2021

Consider a theory which involves a complex scalar ϕ, a Dirac fermion ψ, and a photon field A_μ given by the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D\phi|^2 - m_\phi^2|\phi|^2 + \bar{\psi}\left(i\not{D} - m\right)\psi, \tag{1}$$

where ϕ is a complex scalar field, $F_{\mu\nu}$ is the Maxwell field strength, and D_μ is the covariant derivative defined as

$$D_\mu = \partial_\mu + ieA_\mu. \tag{2}$$

- The complex scalar field ϕ is defined as

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left\{a_{\vec{p}}\, \mathrm{e}^{-ip^\mu x_\mu} + b^\dagger_{\vec{p}}\, \mathrm{e}^{ip^\mu x_\mu}\right\}, \tag{3}$$

  with the commutation relation:

$$[a_{\vec{p}}, a^\dagger_{\vec{k}}] = [b_{\vec{p}}, b^\dagger_{\vec{k}}] = (2\pi)^3\delta^{(3)}(\vec{p} - \vec{k}). \tag{4}$$

- The Maxwell field tensor is defined as $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$, where the expression of the photon field is given by

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_\lambda \left\{\epsilon_{\mu,\lambda}(p)a_{\vec{p},\lambda}\, \mathrm{e}^{-ip^\mu x_\mu} + \epsilon^*_{\mu,\lambda}(p)a^\dagger_{\vec{p}}\, \mathrm{e}^{ip^\mu x_\mu}\right\}. \tag{5}$$

- The Dirac fermion field ψ is defined as follows

$$\psi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_s \left\{b^s_{\vec{p}}u^s(p)\mathrm{e}^{-ip^\mu x_\mu} + c^{s\dagger}_{\vec{p}}v^s(p)\mathrm{e}^{ip^\mu x_\mu}\right\}, \tag{6}$$

  with the annihilation and creation operators satisfying the following anticommutation relation:

$$\{b^s_{\vec{p}}, b^{r\dagger}_{\vec{k}}\} = \{c^s_{\vec{p}}, c^{r\dagger}_{\vec{k}}\} = (2\pi)^3\delta^{sr}\delta^{(3)}(\vec{p} - \vec{k}). \tag{7}$$

(1) The energy-momentum tensor for this theory is given by:

$$\begin{aligned} T^\mu_{\ \nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\partial_\nu\phi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\nu\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\nu A_\lambda - \delta^\mu_{\ \nu}\mathcal{L} &(8)\\ &= \left(\partial^\mu - ieA^\mu\right)\phi^*\partial_\nu\phi + \left(\partial^\mu + ieA^\mu\right)\phi\partial_\nu\phi^* + i\bar{\psi}\gamma^\mu\partial_\nu\psi - F^{\mu\lambda}\partial_\nu A_\lambda - \delta^\mu_{\ \nu}\mathcal{L}. &(9)\end{aligned}$$

The Hamiltonian density $\mathcal{H} = T^{00}$ is then given by:

$$\begin{aligned} \mathcal{H} &= 2\dot{\phi}\dot{\phi}^* + ieA^0\left(\phi\dot{\phi}^* - \phi^*\dot{\phi}\right) + i\psi^\dagger\dot{\psi} - F^{0i}\dot{A}_i - g^{00}\mathcal{L} &(10)\\ &= \dot{\phi}\dot{\phi}^* + (\vec{\nabla}\phi^*)\cdot(\vec{\nabla}\phi) + ie\left[\phi(\vec{\nabla}\phi^*)\cdot\vec{A} - \phi^*(\vec{\nabla}\phi)\cdot\vec{A}\right] + m\bar{\psi}\psi \\ &\quad + \left(m_\phi^2 - e^2A_\mu A^\mu\right)|\phi|^2 + i\bar{\psi}\vec{\gamma}\cdot(\vec{\nabla}\psi) + \frac{1}{2}\left(F^{ij}F^{ij} + F^{0i}F^{0i}\right), &(11)\end{aligned}$$

where the last step holds assuming $A^0 = 0$.

The canonical momenta are respectively for the scalar field $\pi = \dot{\phi}$, for the fermion field $\Pi = \psi^\dagger$ and for the Maxwell field $E^i = F^{0i}$. We thus find

$$\mathcal{H} = \pi\pi^* + (\vec{\nabla}\phi^*)\cdot(\vec{\nabla}\phi) + ie\left[\phi(\vec{\nabla}\phi^*)\cdot\vec{A} - \phi^*(\vec{\nabla}\phi)\cdot\vec{A}\right] + m\bar{\psi}\psi$$
$$+ \left(m_\phi^2 - e^2 A_\mu A^\mu\right)|\phi|^2 + i\bar{\psi}\vec{\gamma}\cdot(\vec{\nabla}\psi) + \frac{1}{2}\left(\vec{E}\cdot\vec{E} + F^{ij}F^{ij}\right). \tag{12}$$

Note that becaue the Dirac Larangian is first order in time derivatives, the Hamiltonian does not depend on the canonical momentum.
(Full credit is given for simply stating that $E^i = F^{0i}$).

(2) Let us now write down the transformations that leave the Lagrangian invariant and derive the corresponding Noether's currents.

– Charge conservation for scalars
  The Lagrangian is invariant under the transformation:

$$\phi \longrightarrow \phi' = \mathrm{e}^{-i\alpha}\phi. \tag{13}$$

  Since the scalar field is complex, the variations of the fields ϕ and ϕ^* has to be considered as independent. The infinitesimal transformations are:

$$\phi \longrightarrow \phi + \Delta\phi \sim (1 - i\alpha)\phi \Rightarrow \Delta\phi = -i\phi \tag{14}$$
$$\phi^* \longrightarrow \phi^* + \Delta\phi^* \sim (1 + i\alpha)\phi^* \Rightarrow \Delta\phi^* = i\phi^* \tag{15}$$

  Using these results to compute the conserved Noether current associated with the above symmetry, we have:

$$\phi \longrightarrow \phi + \Delta\phi \sim (1 - i\alpha)\phi \Rightarrow \Delta\phi = -i\phi \tag{16}$$
$$\phi^* \longrightarrow \phi^* + \Delta\phi^* \sim (1 + i\alpha)\phi^* \Rightarrow \Delta\phi^* = i\phi^* \tag{17}$$

  Using these results to compute the conserved Noether current associated with the above symmetry, we have:

$$\mathcal{J}^\mu_\phi = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\Delta\phi^* \tag{18}$$
$$= (\partial^\mu\phi^* - ieA^\mu\phi^*)(-i\phi) + (\partial^\mu\phi + ieA^\mu\phi)(i\phi^*) \tag{19}$$
$$= i(\phi^* D^\mu\phi - \phi(D^\mu\phi)^*). \tag{20}$$

– Charge conservation for fermions
  For the fermion field, the following transformation leaves the Lagrangian invariant:

$$\psi \longrightarrow \psi' = \mathrm{e}^{-iq}\psi. \tag{21}$$

  The corresponding Noether's current can be written as:

$$\mathcal{J}^\mu_\psi = \frac{\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\psi)}\Delta\psi \qquad \text{with} \qquad \frac{\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\psi)} = i\bar{\psi}\gamma^\mu, \tag{22}$$

and $\Delta\psi$ can be known by considering an infinitesimal transformation

$$\psi \longrightarrow \psi + \Delta\psi, \qquad \Delta\psi = -iq\psi. \tag{23}$$

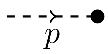
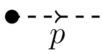
Combining this result to the ones in Eq. (22), we have

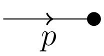
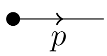
$$\mathcal{J}^{\mu}_{\psi} = q\bar{\psi}\gamma^{\mu}\psi. \tag{24}$$

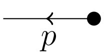
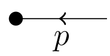
The symmetry transformations are given by the $U(1)$ groups which correspond to the conservation of charges. Hence, the global symmetry group that governs the theory is $U(1) \otimes U(1)$.

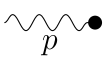
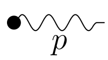
(3) Let us now write down the Feynman rules of the theory.

(a) External lines:

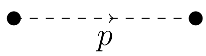
$\phi|s(p)\rangle =$  $= 1,$ $\qquad \langle s(p)|\phi =$  $= 1,$

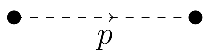
$\psi|f(p,s)\rangle =$  $= u_s(p),$ $\qquad \langle f(p,s)|\bar{\psi} =$  $= \bar{u}_s(p),$

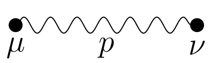
$\bar{\psi}|\bar{f}(p,s)\rangle =$  $= \bar{v}_s(p),$ $\qquad \langle \bar{f}(p,s)|\psi =$  $= v_s(p),$

$A_\mu|\gamma(p,s)\rangle =$  $= \epsilon_\mu(p,s),$ $\qquad \langle \gamma(p,s)|A_\mu =$  $= \epsilon^*_\mu(p,s).$ (25)

(b) Propagators:

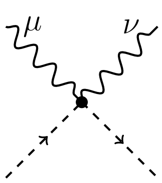
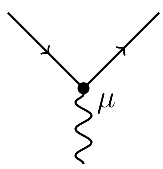
Scalar :  $= \dfrac{i}{p^2 - m^2 + i\epsilon}$ (26)

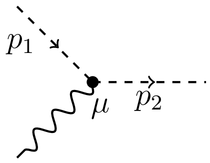
Fermion :  $= \dfrac{i}{p - M^2 + i\epsilon}$ (27)

Photon :  $= -\dfrac{i}{p^2 + i\epsilon}\left[g_{\mu\nu} - (1-\xi)\dfrac{p_\mu p_\nu}{p^2}\right]$ (28)

In the expression of the photon propagator, ξ parametrizes a set of covariant gauges (for a Feynman gauge, $\xi = 1$).

(c) Vertices:

 $= 2ie^2 g^{\mu\nu},$ $\qquad$  $= -ie\gamma^{\mu},$



$$= -ie(p_1 + p_2)^{\mu}, \tag{29}$$

(4) Let us now compute the scattering amplitude of the process $f\bar{f} \to \phi\bar{\phi}$. As shown in Fig. 1, only one single diagram contribute to such a process at leading order. Using the above Feynman rules, the amplitude can be written as:

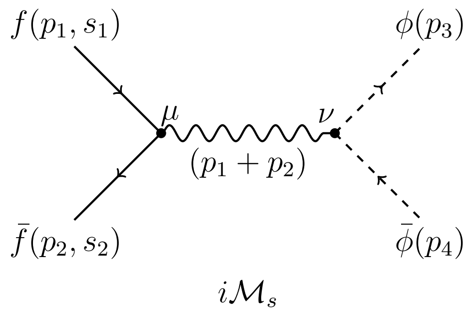


Figure 1: Leading non-vanishing diagrams for $f\bar{f} \longrightarrow \phi\bar{\phi}$.

$$i\mathcal{M}_s = (-ie)^2 \left(p_3 - p_4\right)^{\nu} \left(-\frac{ig_{\mu\nu}}{(p_1 + p_2)^2}\right) u(p_1)\gamma^{\mu}\bar{v}(p_2) \tag{30}$$

$$= i\frac{e^2}{s} \left(p_3 - p_4\right)^{\nu} u(p_1)\gamma_{\nu}\bar{v}(p_2). \tag{31}$$

Taking the modulus square of the above amplitude leads to the following:

$$|\mathcal{M}_s|^2 = \frac{e^4}{s^2} \left(p_3 - p_4\right)^{\mu} \left(p_3 - p_4\right)^{\nu} \mathrm{Tr}\left[u(p_1)\gamma_{\mu}\bar{v}(p_2)v(p_2)\gamma_{\nu}\bar{u}(p_1)\right]. \tag{32}$$

Since we are considering an unpolarized process, we need to average over the polarization of the initial state fermions. This yields:

$$|\bar{\mathcal{M}_s}|^2 = \frac{1}{4}\sum_{s_1,s_2} |\mathcal{M}|^2 \tag{33}$$

$$= \frac{e^4}{4s^2} \left(p_3 - p_4\right)^{\mu} \left(p_3 - p_4\right)^{\nu} \mathrm{Tr}\left[\left(\not{p}_1 + m\right)\gamma_{\mu}\left(\not{p}_2 - m\right)\gamma_{\nu}\right], \tag{34}$$

where to go from the first to the second line we used the completeness relations $\sum_{s_1} \bar{u}(p_1, s_1)u(p_1, s_1) = (\not{p}_1 + m)$ and $\sum_{s_2} \bar{v}(p_2, s_2)v(p_2, s_2) = (\not{p}_2 - m)$.
The expression of the amplitude in Eq. (34) can be simplified using the following property of trace

$$\mathrm{Tr}\left[\left(\not{p} + m\right)\gamma_{\mu}\left(\not{k} - m\right)\gamma_{\nu}\right] = 4\left[p_{\mu}k_{\nu} + p_{\nu}k_{\mu} - g_{\mu\nu}m^2 - g_{\mu\nu}(pk)\right], \tag{35}$$

where the parenthesis denote the usual scalar product. Thus, Eq.(34) now becomes:

$$|\bar{\mathcal{M}_s}|^2 = \frac{e^4}{s^2} \left(p_3 - p_4\right)^{\mu} \left(p_3 - p_4\right)^{\nu} \left[(p_1)_{\mu}(p_2)_{\nu} + (p_1)_{\nu}(p_2)_{\mu} - g_{\mu\nu}(m^2 + (p_1p_2))\right] \tag{36}$$

$$= \frac{e^4}{s^2}\left[2(p_1p_3)(p_2p_3) - p_3^2(m^2 + (p_1p_2))\right] + \frac{e^4}{s^2}\left[2(p_1p_4)(p_2p_4) - p_4^2(m^2 + (p_1p_2))\right]$$
$$+ \frac{2e^4}{s^2}\left[(p_1p_3)(p_2p_4) + (p_1p_4)(p_2p_3) - (p_3p_4)(m^2 + (p_1p_2))\right] \tag{37}$$

The above expression can be simplified further using the following definition of the Mandesltam variables

$$(p_1p_2) = \frac{s}{2} - m^2 \tag{38}$$
$$(p_3p_4) = \frac{s}{2} - m_\phi^2 \tag{39}$$
$$(p_1p_3) = (p_2p_4) = \frac{1}{2}\left(m^2 + m_\phi^2 - t\right) \tag{40}$$
$$(p_1p_4) = (p_2p_3) = \frac{1}{2}\left(m^2 + m_\phi^2 - u\right), \tag{41}$$

Using the above definitions and performing some algebraic simplifications, the amplitude in Eq. (34) now becomes:

$$|\bar{\mathcal{M}}_s|^2 = \frac{e^4}{2s^2}\left[s^2 - 4m_\phi^2 s - (t-u)^2\right]. \tag{42}$$

(5) Let us compute the high-energy limit of the amplitude computed in the previous section. In the high energy limit $\frac{m^2}{s} \to 0$ and $\frac{m_\phi^2}{s} \to 0$ so the limit is found setting the masses to zero ($m = m_\phi = 0$) in Eq. (42), and the amplitude in this limit is given by

$$|\bar{\mathcal{M}}_s^{\mathrm{HE}}|^2 = \frac{e^4}{2s^2}\left[s^2 - (t-u)^2\right]. \tag{43}$$

Now, let us compute the minimal energy (say for the incoming fermion p_1) that is required in order for the above process to take place. Let us work in the center of mass system (CMS) where $\vec{p_1} = -\vec{p_2} = \vec{p}$. The total CMS energy of the scalars after the collision is given by:

$$s = E_{\mathrm{Out}}^2 = \left(\sqrt{m_\phi^2 + |\vec{p}'|^2} + \sqrt{m_\phi^2 + |\vec{p}'|^2}\right)^2, \tag{44}$$

so the minimum s_{min} corresponds to the two scalars being at rest in the CMS, i.e. $\vec{p}' = 0$ and

$$s_{\mathrm{min}} = 4m_\phi^2. \tag{45}$$

But

$$s_{\mathrm{min}} = (p_s + p_2)^2 = 4E^2 = 4\sqrt{m^2 + \vec{p}^2}, \tag{46}$$

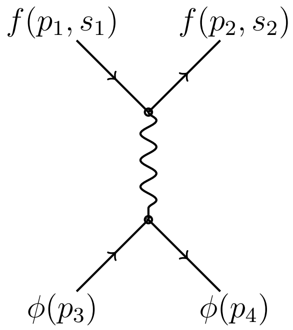
where $\vec{p}$ is the three-momentum of the incoming particles. Hence if $m^2 \geq m_\phi^2$ the process always happens, if $m^2 < m_\phi^2$ the center-of-mass energy of the incoming particles must be at least as given by Eq. (45).

We can use the result form Eq. (45) to derive the value of the amplitude corresponding to the minimum energy in the center of mass frame. As mentioned before, the minimum energy corresponds to the two scalars being at rest in the CMS. This immediately implies that $(p_1 - p_3) = (p_1 - p_4)$, so $t = u$. Hence, we have

$$|\bar{\mathcal{M}}_s^{\mathrm{Em}}|^2 = |\bar{\mathcal{M}}_s|^2 = \frac{e^4}{2s_{\mathrm{min}}^2}\left[s_{\mathrm{min}}^2 - 4m_\phi^2 s_{\mathrm{min}}\right] = 0. \tag{47}$$

(6) Let us now derive the modulus square of the process $f\phi \to f\phi$. At leading order, only one single diagram contribute:

$$i\mathcal{M}_t(f\phi \to f\phi) \equiv \tag{48}$$



Using the Feynman rules in Sec. (3), we can write down the expression of the scattering amplitude:

$$\mathcal{M}_t = \frac{e^2}{(p_1 - p_2)^2}\,(p_3 + p_4)^\mu\,\bar{u}(p_2)\gamma_\mu u(p_1). \tag{49}$$

Multiplying Eq. (49) by its complex conjugate and using the completeness relations, we find that the modulus square of the amplitude can be written as:

$$|\mathcal{M}_t|^2 = \frac{e^4}{(p_1 - p_2)^4}\,(p_3 + p_4)^\mu\,(p_3 + p_4)^\nu\,\mathrm{Tr}\left[\left(\not{p}_1 + m\right)\gamma_\mu\left(\not{p}_2 + m\right)\gamma_\nu\right]. \tag{50}$$

We can notice that Eq. (50) is related to Eq. (32) with the following transformation of the momenta:

$$|\mathcal{M}_t|^2 = -|\mathcal{M}_s\left(p_1 \to p_1, p_2 \to -p_2, p_3 \to p_4, p_4 \to -p_3\right)|^2. \tag{51}$$

In terms of Mandelstam variables, this can be expressed as:

$$|\bar{\mathcal{M}}_t|^2 = -|\bar{\mathcal{M}}_s\left(s \to t, t \to u, u \to s\right)|^2. \tag{52}$$

Therefore, using Eq. (42), we can derive the modulus square for the unpolarized process $f\phi \to f\phi$ which reads as:

$$|\bar{\mathcal{M}}_t|^2 = \frac{e^4}{2t^2}\left[(u - s)^2 + 4m_\phi^2 t - t^2\right]. \tag{53}$$