Solution of the exam of Theoretical Physics of July 19 2023

Electromagnetic field:

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=\pm} \left(a_p^\lambda \epsilon_\mu(p,\lambda) e^{-ipx} + a_p^{\lambda\dagger} \epsilon_\mu^*(p,\lambda) e^{ipx} \right) \tag{1}$$

Spinor field:

$$\psi_i(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1}^{2} \left(a_{i,p}^s u_i^s(p) e^{-ipx} + b_{i,p}^{s\dagger} v_i^s(p) e^{ipx} \right) \quad i = 1,2 \tag{2}$$

1. The energy-momentum tensor is defined as

$$T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_1)}\partial_\nu\psi_1 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_2)}\partial_\nu\psi_2 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\nu A_\lambda - \delta^\mu{}_\nu\mathcal{L}. \tag{3}$$

   Using

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)} = i\bar{\psi}_i\gamma^\mu, \tag{4}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)} = -F^{\mu\lambda}, \tag{5}$$

   we find

$$T^\mu{}_\nu = i\bar{\psi}_1\gamma^\mu\partial_\nu\psi_1 + i\bar{\psi}_2\gamma^\mu\partial_\nu\psi_2 - F^{\mu\lambda}\partial_\nu A_\lambda - \delta^\mu{}_\nu\mathcal{L}. \tag{6}$$

   The hamiltonian density is

$$\mathcal{H} = T^{00} = i\bar{\psi}_1\gamma^0\partial_0\psi_1 + i\bar{\psi}_2\gamma^0\partial_0\psi_2 - F^{0\lambda}\partial^0 A_\lambda - \mathcal{L} \tag{7}$$

$$= i\psi_1^\dagger\dot{\psi}_1 + i\psi_2^\dagger\dot{\psi}_2 - F^{0i}\dot{A}_i - \left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}_1\left(i\not{D} - m_1\right)\psi_1 + \bar{\psi}_2\left(i\not{D} - m_2\right)\psi_2 \right]. \tag{8}$$

   In the Coulomb gauge $A^0 = 0$ and therefore $\dot{A}_i = F^0{}_i = -F^{0i}$. Moreover we have $D^0 = \partial^0$. We thus get the simplified expression

$$\mathcal{H} = \frac{1}{2}\left(F^{0i}F^{0i} + \frac{1}{2}F^{ij}F^{ij} \right) - \bar{\psi}_1\left(i\vec{\gamma}\cdot\left(\vec{\nabla} - ie\vec{A}\right) - m_1 \right)\psi_1 - \bar{\psi}_2\left(i\vec{\gamma}\cdot\left(\vec{\nabla} - ie\vec{A}\right) - m_2 \right)\psi_2. \tag{9}$$

   Note that we have assumed that $\vec{\nabla}$ denotes ∂_i, but $\vec{A}$ denotes $A^i = -A_i$.

2. The symmetries of the theory are:

$$\begin{aligned} \psi_1 &\to \psi_1' = e^{-i\theta_1}\psi_1, \\ \psi_2 &\to \psi_2' = e^{-i\theta_2}\psi_2 \end{aligned} \tag{10}$$

   that correspond to a $U(1)_1 \otimes U(1)_2$ symmetry. The Noether currents are

$$\mathcal{J}_1^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_1)}\Delta\psi_1 = \bar{\psi}_1\gamma^\mu\psi_1, \tag{11}$$

$$\mathcal{J}_2^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_2)}\Delta\psi_2 = \bar{\psi}_2\gamma^\mu\psi_2, \tag{12}$$

   where we used

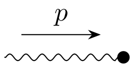
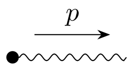
$$\Delta\psi_i = \frac{\delta\psi_i}{\theta_i} = \frac{-i\theta_i\psi_i}{\theta_i} = -i\psi_i. \tag{13}$$

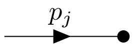
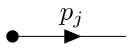
   The charge operators are

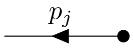
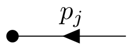
$$Q_i = \int d^3x J_i^0(x) = \int d^3x\, \psi_i^\dagger(x)\psi_i(x) = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} \left(a_{i,p}^{s\dagger}a_{i,p}^s - b_{i,p}^{s\dagger}b_{i,p}^s \right). \tag{14}$$

   The derivation can be found in standard textbooks. The conservation law corresponds to fermion number conservation for each of the two fermion species independently .

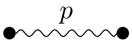
- **External lines**

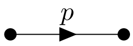
$$A_\mu \left|\gamma(p,\lambda)\right\rangle =$$  $$= \epsilon(p,\lambda), \qquad \left\langle\gamma(p,\lambda\right| A_\mu =$$  $$= \epsilon^*(p,\lambda),$$

$$\psi_i \left|f_i(p_j,s)\right\rangle =$$  $$= u_i^s(p_j), \qquad \left\langle f_i(p_j,s)\right| \bar{\psi}_i =$$  $$= \bar{u}_i^s(p_j),$$

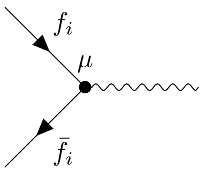
$$\bar{\psi}_i \left|\bar{f}_i(p_j,s)\right\rangle =$$  $$= \bar{v}_i^s(p_j), \qquad \left\langle f_i(p_j,s)\right| \bar{\psi}_i =$$  $$= v_i^s(p_j),$$

- **Propagators**

μ  ν $$= -\frac{i}{p^2+i\epsilon}\left(g_{\mu\nu}-(1-\xi)\frac{p_\mu p_\nu}{p^2}\right), \quad \text{photon propagator}$$

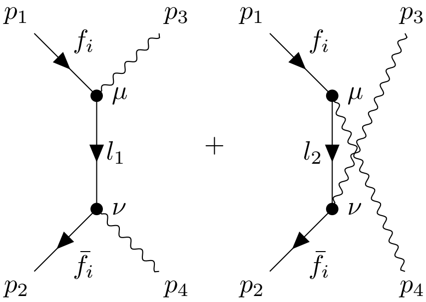
 $$= \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}, \quad \text{fermion propagator}$$

- **Vertex**



$$= -ie\gamma^\mu.$$

3. The diagrams that contribute to the process are



$$= i\mathcal{M}_i^{(t)} + i\mathcal{M}_i^{(u)} = i\mathcal{M}_i \tag{15}$$

with $l_1 = p_1 - p_3$ and $l_2 = p_1 - p_4$.

4. The amplitudes are

$$i\mathcal{M}_i^{(t)} = \bar{v}_i^{s_2}(p_2)(-ie\gamma^\nu)\frac{i(\not{l}_1+m)}{l_1^2-m^2}(-ie\gamma^\mu)u_i^{s_1}(p_1)\epsilon_\nu^*(p_4,\lambda_4)\epsilon_\mu^*(p_3,\lambda_3), \tag{16}$$

$$i\mathcal{M}_i^{(u)} = \bar{v}_i^{s_2}(p_2)(-ie\gamma^\nu)\frac{i(\not{l}_2+m)}{l_2^2-m^2}(-ie\gamma^\mu)u_i^{s_1}(p_1)\epsilon_\nu^*(p_3,\lambda_3)\epsilon_\mu^*(p_4,\lambda_4), \tag{17}$$

where the fermion and antifermion in the initial state must have the same mass, and $m = m_1$ or $m = m_2$ according to the nature of the incoming fermion.

The unpolarized amplitude is obtained averaging over the spins of the initial fermions and summing over the polarizations of the final photons, using the identities

$$\sum_{s=1,2} u_p^s \bar{u}_p^s = \not{p} + m \qquad \sum_{s=1,2} v_p^s \bar{v}_p^s = \not{p} - m \tag{18}$$

and performing the sum over photon polarization as suggested in the assignment.

Setting $m_i = 0$ the result becomes independent of the species of incoming fermion. So, regardless of the nature of the incoming fermion pair we get

$$|\bar{\mathcal{M}}_i|^2 = \frac{1}{4}\sum_{s_1,s_2}\sum_{\lambda_3,\lambda_4}|\mathcal{M}_i|^2 \tag{19}$$

$$= \frac{e^4}{4}\left\{\frac{1}{l_1^4}\mathrm{Tr}\left[\not{p}_2\gamma^\nu\not{l}_1\gamma^\mu\not{p}_1\gamma^{\mu'}\not{l}_1\gamma^{\nu'}\right]g_{\mu\mu'}g_{\nu\nu'} + \frac{1}{l_2^4}\mathrm{Tr}\left[\not{p}_2\gamma^\nu\not{l}_2\gamma^\mu\not{p}_1\gamma^{\mu'}\not{l}_2\gamma^{\nu'}\right]g_{\nu\nu'}g_{\mu\mu'}\right. \tag{20}$$

$$\left. + \frac{1}{l_1^2 l_2^2}\mathrm{Tr}\left[\not{p}_2\gamma^\nu\not{l}_1\gamma^\mu\not{p}_1\gamma^{\mu'}\not{l}_2\gamma^{\nu'}\right]g_{\mu\nu'}g_{\nu\mu'} + \frac{1}{l_1^2 l_2^2}\mathrm{Tr}\left[\not{p}_2\gamma^\nu\not{l}_2\gamma^\mu\not{p}_1\gamma^{\mu'}\not{l}_1\gamma^{\nu'}\right]g_{\mu\nu'}g_{\nu\mu'}\right\}. \tag{21}$$

Using the identities

$$\begin{aligned}\gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu &= 4g^{\alpha\beta}\\ \gamma^\mu\gamma^\alpha\gamma_\mu &= -2\gamma^\alpha\\ \gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\delta\gamma_\mu &= -2\gamma^\delta\gamma^\beta\gamma^\alpha\end{aligned} \tag{22}$$

we get

$$|\bar{\mathcal{M}}_i|^2 = 8e^4\left[\frac{1}{l_1^2}(p_1\cdot p_3)(p_2\cdot p_3) + \frac{1}{l_2^2}(p_1\cdot p_4)(p_2\cdot p_4) - 2(p_1\cdot p_2)\Big((p_3\cdot p_4) - (p_1\cdot p_3) - (p_1\cdot p_4)\Big)\right]. \tag{23}$$

The last term is seen to vanish in the massless case: indeed, using momentum conservation

$$p_2 = p_3 + p_4 - p_1 \tag{24}$$

so, squaring both sides of the equation and using $p_i^2 = 0$ for all i,

$$0 = p_3\cdot p_4 - p_1\cdot p_3 - p_1\cdot p_4. \tag{25}$$

We thus find

$$|\bar{\mathcal{M}}_i|^2 = 8e^4\left[\frac{(p_1\cdot p_3)(p_2\cdot p_3)}{(p_1-p_3)^2} + \frac{(p_1\cdot p_4)(p_2\cdot p_4)}{(p_1-p_4)^2}\right]. \tag{26}$$

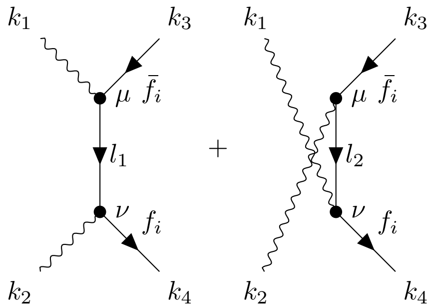
5. Defining the Mandelstam variables as

$$\begin{aligned}s &= (p_1+p_2)^2 = (p_3+p_4)^2 = 2(p_1\cdot p_2) = 2(p_3\cdot p_4),\\ t &= l_1^2 = (p_1-p_3)^2 = (p_4-p_2)^2 = -2(p_1\cdot p_3) = -2(p_2\cdot p_4),\\ u &= l_2^2 = (p_1-p_4)^2 = (p_3-p_2)^2 = -2(p_1\cdot p_4) = -2(p_2\cdot p_3),\end{aligned} \tag{27}$$

we find

$$|\bar{\mathcal{M}}_i|^2 = 2e^4\left(\frac{u}{t} + \frac{t}{u}\right). \tag{28}$$

6. The diagrams that contribute to the process are the following



$$= \; i\mathcal{M}_i^{(t),\text{cross}} + i\mathcal{M}_i^{(u),\text{cross}} = i\mathcal{M}_i^{\text{cross}}, \tag{29}$$

with $l_1 = k_1 - k_3$ and $l_2 = k_3 - k_2$. It is apparent that the amplitudes can be obtained from those computed at point (5) by performing the substitutions

$$p_1 \to -k_3 \tag{30}$$

$$p_2 \to -k_4 \tag{31}$$

$$p_3 \to -k_1 \tag{32}$$

$$p_4 \to -k_2, \tag{33}$$

so that

$$i\mathcal{M}\Big\{\gamma(k_1) + \gamma(k_2) \to f_i(k_4) + \bar{f}_i(k_3)\Big\} = i\mathcal{M}\Big\{\bar{f}_i(-k_4) + f_i(-k_3) \to \gamma(-k_1) + \gamma(-k_2)\Big\} \tag{34}$$

(crossing symmetry).

This implies

$$t \to t, \quad u \to u \tag{35}$$

Hence

$$i\mathcal{M}_i^{\text{cross}} = i\mathcal{M}_i \tag{36}$$

However, if we cannot distinguish the nature of the fermions we must keep into account the fact that at point (5) the fermions were incoming (so the result was the same independent of which fermion enters the process) while now the fermions are outgoing (so we must sum over the possible final states). The reason is the same why we average over intial polarizaton but sum over final ones.

Hence, the result is now multiplied by a factor 2 in comparison to that of point (6):

$$|\bar{\mathcal{M}}^{\text{cross}}|^2 = \sum_i |\bar{\mathcal{M}}_i^{\text{cross}}|^2 = 4e^4 \left(\frac{u}{t} + \frac{t}{u} \right) \tag{37}$$

7. In the case $m_1 = m_2 = 0$ the Dirac part of the lagrangian can be written as

$$\begin{aligned} \mathcal{L}_D &= \bar{\psi}_1 i\not{D}\psi_1 + \bar{\psi}_2 i\not{D}\psi_2 = \bar{\psi}_{1,L} i\not{D}\psi_{1,L} + \bar{\psi}_{1,R} i\not{D}\psi_{1,R} + \bar{\psi}_{2,L} i\not{D}\psi_{2,L} + \bar{\psi}_{2,R} i\not{D}\psi_{2,R} &(38)\\ &= \bar{\Psi}_L i\not{D}\Psi_L + \bar{\Psi}_R i\not{D}\Psi_R, &(39) \end{aligned}$$

where we defined

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{40}$$

The symmetry now is

$$\Psi_L \to \Psi'_L = U_L \Psi_L\,, \quad \Psi_R \to \Psi'_R = U_R \Psi_R, \tag{41}$$

where U_L and U_R are generic 2×2 unitary matrices, which in turn can be written as a 2×2 unitary matrix with determinant equal to 1 (special unitary matrix) times a phase transformation.

Hence the symmetry is $U(2)_R \otimes U(2)_L = SU(2)_R \otimes U(1)_R \otimes SU(2)_L \otimes U(1)_L$ symmetry.

Full score is given for noting that U_i are generic 2×2 unitary matrices.

The Noether currents are

$$\begin{aligned} U(1)_{R,L}:& \quad \mathcal{J}^\mu_{R,L} = \bar{\Psi}_{R,L}\gamma^\mu \Psi_{R,L}, \\ SU(2)_{R,L}:& \quad \mathcal{J}^{a,\mu}_{R,L} = \bar{\Psi}_{R,L}\sigma_a\gamma^\mu \Psi_{R,L} \end{aligned} \tag{42}$$

and the charges are

$$\begin{aligned} U(1)_{R,L}:& \quad Q_{R,L} = \int d^3x\, \Psi^\dagger_{R,L}\Psi_{R,L}, \\ SU(2)_{R,L}:& \quad Q^a_{R,L} = \int d^3x\, \Psi^\dagger_{R,L}\sigma_a \Psi_{R,L}, \end{aligned} \tag{43}$$

where σ_i are Pauli matrices.

The commutation relations are

$$[Q_{R,L}, Q_{L,R}] = [Q_{R,L}, Q^a_{L,R}] = [Q^a_{R,L}, Q^b_{L,R}] = 0. \tag{44}$$

$$[Q^a_{R,L}, Q^b_{R,L}] = i\epsilon_{abc} Q^c_{R,L}. \tag{45}$$

Full score is given for just stating that the commutation relations between the charges are the same as the commutation relations of the generators of the unitary transformations.