SOLUTIONS TO ESAME SCRITTO DI FISICA TEORICA I

2 ore e 30 minuti; non sono consentiti libri o appunti

11 luglio 2025

1. The energy-momentum tensor is

$$T^{\mu\nu} = \sum_i^2 \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial^\nu \phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i^*)} \partial^\nu \phi_i^* \right] - \eta^{\mu\nu} \mathcal{L} \tag{1}$$

$$= \sum_i \left[(D^\mu \phi_i)^* \partial^\nu \phi_i + (D^\mu \phi_i) \partial^\nu \phi_i^* \right] - \eta^{\mu\nu} \mathcal{L}. \tag{2}$$

The Hamiltonian density is the 00 component of this, so, not including the contribution from the Maxwell tensor, we have

$$\mathcal{H} = 2 \sum_i \dot{\phi}_i \dot{\phi}_i^* + ieA^0 \left(\phi_i \dot{\phi}_i^* - \phi_i^* \dot{\phi}_i \right) - g^{00} \mathcal{L}_\phi \tag{3}$$

$$= \sum_i |\dot{\phi}_i|^2 - e^2 A_0^2 |\phi_i|^2 + (\vec{D}\phi_i)^* \cdot \vec{D}\phi_i + m_i^2 |\phi_i|^2. \tag{4}$$

2. The global symmetry is

$$\phi_i \to \phi_i' = e^{i\theta_i} \phi_i, \tag{5}$$

where the two phases can be different, so the global symmetry is U(1)×U(1). The infinitesimal transformation of the i-th field is

$$\delta \phi_i = i\theta_i \phi_i \tag{6}$$

$$\delta \phi_i^* = -i\theta_i \phi_i^*. \tag{7}$$

There are two conserved Noether currents, given by

$$j_i^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \frac{\delta \phi_i}{\theta_i} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i^*)} \frac{\delta \phi_i^*}{\theta_i} \tag{8}$$

$$= \left[i(D^\mu \phi_i)^* \phi_i - i(D^\mu \phi_i) \phi_i^* \right]. \tag{9}$$

$$\tag{10}$$

There are correspondingly two conserved charges $Q_i = \int d^3x \; j_i^0(x)$, and we use the following mode expansions for the fields and their derivatives:

$$\phi_i = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{E_p}} (a_i(p) e^{-ipx} + b_i^\dagger(p) e^{ipx}) \tag{11}$$

$$\partial_\mu \phi_i = \int \frac{d^3p}{(2\pi)^3} \frac{ip_\mu}{\sqrt{E_p}} (-a_i(p) e^{-ipx} + b_i^\dagger(p) e^{ipx}), \tag{12}$$

and the conjugate fields of this. To calculate the charge in terms of ladder operators, we have to calculate the position integrals of the terms in the Noether current, using the mode expansions of the fields. We get

$$Q_i = \int \frac{d^3p}{(2\pi)^3} \left(a_i^\dagger a_i - b_i^\dagger b_i \right). \tag{13}$$

The physical meaning is that the charges of the two species of scalar fields are separately conserved.

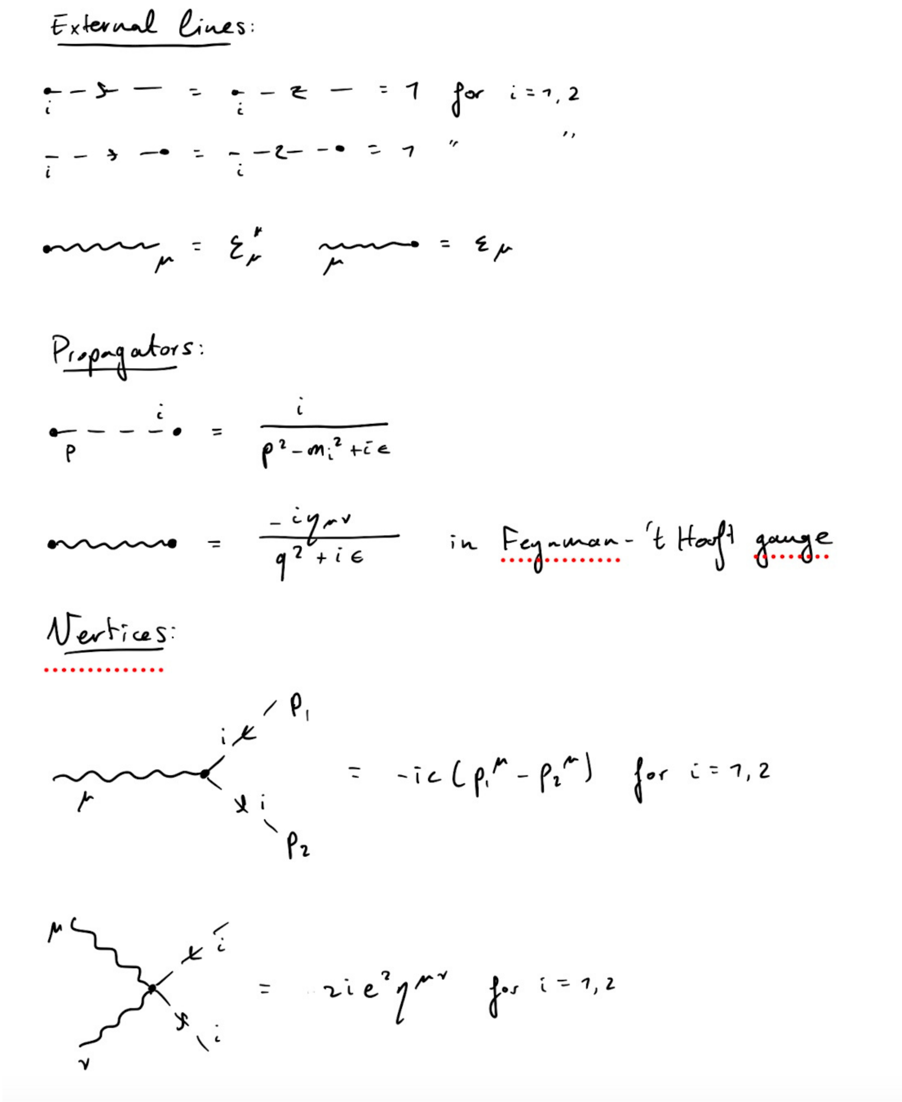

Figure 1: the Feynman rules for this Lagrangian

3. The Feynman rules are given in Fig. 1
4. We have an s-channel for process (1) and a t-channel for process (2), both drawn in Fig. 2

   For process (1), the amplitude is written as follows:

$$M_s^{(1)} = -e^2(p_1^\mu - p_2^\mu)\frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2}(p_4^\nu - p_3^\nu), \tag{14}$$

   the only subtlety being the minus sign between the momenta, for which we can look at the mode expansions of the fields to know what sign it should be. The other amplitude is given by:

$$M_t^{(2)} = -e^2({p'_1}^\mu + {p'_3}^\mu)\frac{-i\eta_{\mu\nu}}{(p'_3-p'_1)^2}({p'_2}^\nu + {p'_4}^\nu). \tag{15}$$

   Comparing the two results one sees that the second amplitude can be obtained from the first by letting $p'_3 = -p_2$ and $p'_2 = -p_3$, i.e. replacing the momentum of the incoming antiparticles with minus the momentum of the outgoing particle, and the momentum of the outgoing antiparticles with minus the momentum of the incoming particle.
5. The squared amplitudes are

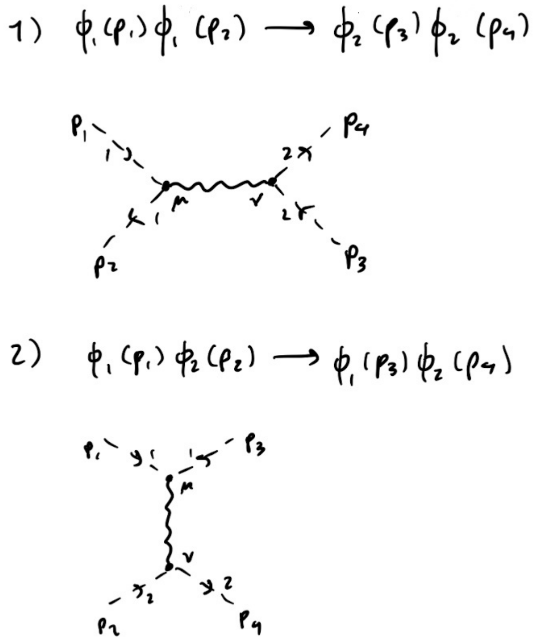

Figure 2: Feynman diagrams for the two processes

$$
\begin{aligned}
|M_s|^2_{(1)} &= \frac{e^4}{s^2}(p_2^\mu - p_1^\mu)(p_{3,\mu} - p_{4,\mu}) \cdot (p_2^\nu - p_1^\nu)(p_{3,\nu} - p_{4,\nu}) \\
&= \frac{e^4}{s^2}[(p_2 - p_1) \cdot (p_3 - p_4)]^2 \\
&= \frac{e^2}{s^2}[p_2 \cdot p_3 + p_2 \cdot p_4 - p_2 \cdot p_4 - p_1 \cdot p_3]^2; \\
|M_t|^2_{(2)} &= \frac{e^4}{t^2}[(p_1 + p_3) \cdot (p_2 + p_4)]^2 \\
&= \frac{e^4}{t^2}[p_1 \cdot p_2 + p_3 \cdot p_4 + p_1 \cdot p_4 + p_3 \cdot p_2]^2.
\end{aligned} \tag{16}
$$

6. For process (1) we have for the Mandelstams:

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = 2m_1^2 + 2p_1 \cdot p_2 \\
&= (p_3 + p_4)^2 = 2m_2^2 + 2p_3 \cdot p_4; \\
t &= (p_3 - p_1)^2 = m_2^2 + m_1^2 - 2p_3 \cdot p_1 \\
&= (p_4 - p_2)^2 = m_2^2 + m_1^2 - 2p_4 \cdot p_2; \\
u &= (p_4 - p_1)^2 = m_2^2 + m_1^2 - 2p_4 \cdot p_1 \\
&= (p_3 - p_2)^2 = m_2^2 + m_1^2 - 2p_3 \cdot p_2.
\end{aligned} \tag{17, 18}
$$

Using these, we can rewrite the squared amplitude from the previous question as

$$
|M_s|^2_{(1)} = \frac{e^4}{s^2}[t - m_2^2 - m_1^2 - u + m_2^2 + m_1^2]^2 = \frac{e^4}{s^2}(t - u)^2. \tag{19}
$$

For process (2), the Mandelstams have slightly different definitions in terms of mass:

$$
\begin{aligned}
s &= m_1^2 + m_2^2 + 2p_1 \cdot p_2 = m_1^2 + m_2^2 + 2p_3 \cdot p_4; \\
t &= 2m_1^2 - 2p_3 \cdot p_1 = 2m_2^2 - 2p_4 \cdot p_2; \\
u &= m_2^2 + m_1^2 - 2p_4 \cdot p_1 = m_1^2 + m_2^2 - 2p_3 \cdot p_2.
\end{aligned} \tag{20}
$$

The squared amplitude then becomes

$$|M_t|^2_{(2)} = \frac{e^4}{t^2}(s-u)^2 \tag{21}$$

The physical regions in which these processes can take place in the case that $m_1 < m_2$ is found by noting that the initial state of a process must have enough energy to produce the final state. For process (1), the final state is heavier than the initial state, so we have to impose a condition on the CM energy: $p_1 + p_2 = \sqrt{s} \geq 2m_2$.

In the case of process (2), the initial state and the final state have the same mass, so there is no need to impose extra conditions.

7. In this limit, we can neglect all massess i.e. $m_i \approx 0$. This changes the expression of the Mandelstam invariants, but the squared amplitudes expressed in terms of Mandelstam invariants remain the same.

8. In the limit in which we can neglect masses, the lagrangian has an extra symmetry

$$\phi \to \phi' = \exp i\vec{\theta} \cdot \sigma\phi, \tag{22}$$

where we have defined a field doublet

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{23}$$

and σ^i are Pauli matrices. The conserved currents are

$$j^a_\mu = i(\partial^\mu \phi^*_i)\sigma^a_{ij}\phi_j - i(\partial^\mu \phi_i)\sigma^a_{ij}\phi^*_j. \tag{24}$$

This is an extra SU(2) symmetry, to be added to the previous U(1), so the overall symmetry is $U(1) \times SU(2)$.

9. The 1-loop corrections to the propagators are given by the diagrams in Fig. 3. The degree of divergence is found noting that there is one four-dimensional loop integral, and either one or two propagators, and it is thus given by by $D = 4 - 2P$. Furthermore te three-point vertex carries one positive power of momentum. The two diagrams with two propagators and two three-point vertices are thus quadratically divergent, while the two diagrams with one propagator and one four-point vertex are also quadratically divergent. Hence all these diagram are quadratically divergent.

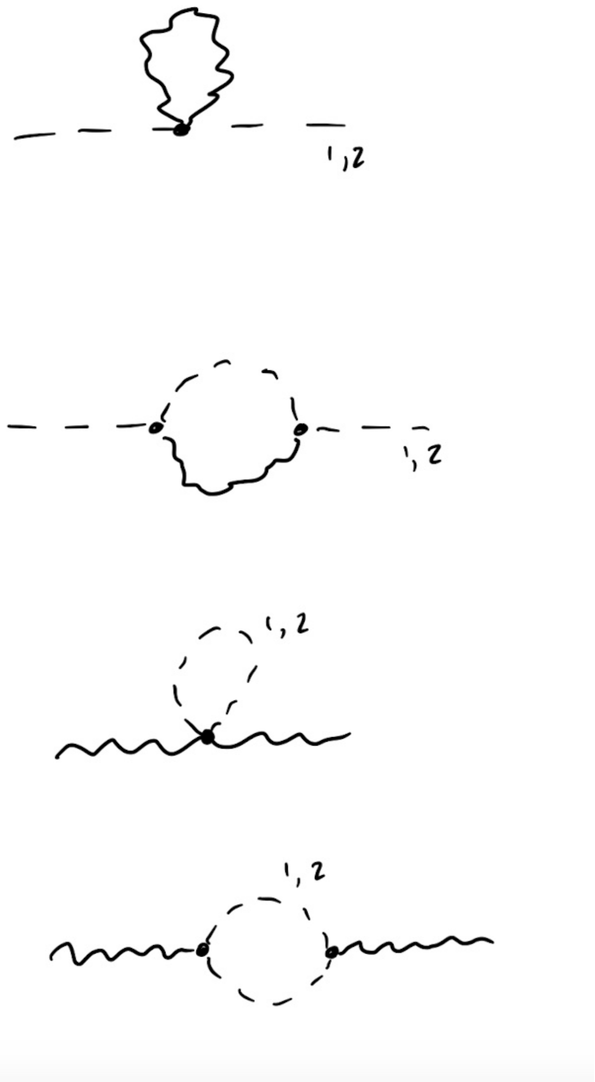

Figure 3: Loop corrections to the two-point functions (i.e. propagators)