Quantum Field Theory I: written test solution

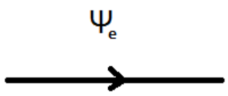
September 27, 2017

1. Consider the following Lagrangian:

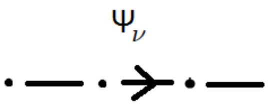
$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_e\left(i\partial\!\!\!/ - m_e\right)\psi_e + \bar{\psi}_\nu i\partial\!\!\!/\psi_\nu + g\bar{\psi}_e\gamma^\mu\left(1-\gamma_5\right)\psi_\nu A_\mu + g\bar{\psi}_\nu\gamma^\mu\left(1-\gamma_5\right)\psi_e A_\mu. \tag{1}$$

The Feynman Rules for this theory are the following:

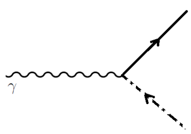
$$\frac{-ig^{\alpha\beta}}{p^2+i\epsilon} \tag{2}$$



$$\frac{i\left(p\!\!\!/ + m_e\right)}{p^2 - m_e^2 + i\epsilon} \tag{3}$$



$$\frac{i}{p\!\!\!/ + i\epsilon} \tag{4}$$



$$ig\gamma^\mu\left(1-\gamma_5\right) \tag{5}$$

2. The process

$$e\left(p_1\right) + \bar{\nu}\left(p_2\right) \to e\left(p_3\right) + \bar{\nu}\left(p_4\right) \tag{6}$$

proceeds at tree level through the single s channel Feynman diagram depicted in Fig. 1. The corresponding matrix element is

$$i\mathcal{M}_s = \frac{ig^2}{s}\bar{u}_e\left(p_3\right)\gamma^\mu\left(1-\gamma_5\right)v_\nu\left(p_4\right)\bar{v}_\nu\left(p_2\right)\gamma_\mu\left(1-\gamma_5\right)u_e\left(p_1\right). \tag{7}$$

The process

$$e\left(p_1\right) + \nu\left(p_2\right) \to e\left(p_3\right) + \nu\left(p_4\right) \tag{8}$$

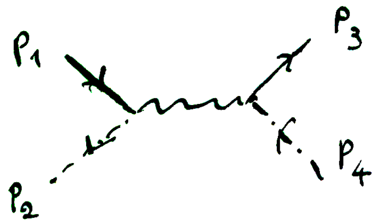


Figure 1: Feynman diagram for the e, $\bar{\nu}$ scattering process

also proceeds at leading order through a single Feynman diagram, the u channel diagram depicted in Fig. 2.

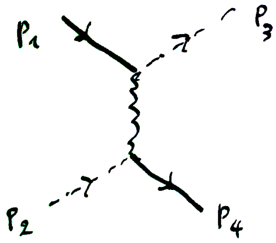


Figure 2: Feynman diagram for the e, ν scattering process

The matrix element is in this case

$$i\mathcal{M}_u = \frac{ig^2}{u}\bar{u}_e(p_3)\gamma^\mu(1-\gamma_5)u_\nu(p_2)\bar{u}_\nu(p_4)\gamma_\mu(1-\gamma_5)u_e(p_1)\,. \tag{9}$$

3. We are now ready to evaluate the square modulus of the matrix element for each process, and to perform the sum over polarizations. For the matrix element Eq. (7), we obtain:

$$\begin{aligned}
\frac{1}{4}\sum|\mathcal{M}|^2 &= \frac{g^4}{4s^2}\bar{u}_e(p_1)\gamma_\nu(1-\gamma_5)v_\nu(p_2)\bar{v}_\nu(p_4)\gamma^\nu(1-\gamma_5)u_e(p_3)\\
&\quad \bar{u}_e(p_3)\gamma^\mu(1-\gamma_5)v_\nu(p_4)\bar{v}_\nu(p_2)\gamma_\mu(1-\gamma_5)u_e(p_1)\\
&= \frac{g^4}{4s^2}\mathrm{Tr}\left[\left(\not{p}_1+m_e\right)\gamma_\nu(1-\gamma_5)\not{p}_2\gamma_\mu(1-\gamma_5)\right]\\
&\quad \mathrm{Tr}\left[\not{p}_4\gamma^\nu(1-\gamma_5)\left(\not{p}_3+m_e\right)\gamma^\mu(1-\gamma_5)\right]\\
&= \frac{16g^4}{s^2}\left[(p_1)_\nu(p_2)_\mu - g_{\mu\nu}(p_1\cdot p_2)+(p_1)_\mu(p_2)_\nu + i\epsilon_{\sigma\nu\rho\mu}p_1^\sigma p_2^\rho\right]\\
&\quad \left[p_4^\nu p_3^\mu - g_{\mu\nu}(p_4\cdot p_3)+p_4^\mu p_3^\nu + i\epsilon^{\eta\nu\lambda\mu}(p_4)_\eta(p_3)_\lambda\right]\\
&= \frac{32g^4}{s^2}\left[(p_1\cdot p_4)(p_2\cdot p_3)+(p_1\cdot p_3)(p_2\cdot p_4)+\left(g^\eta{}_\sigma g^\lambda{}_\rho - g^\lambda{}_\sigma g^\eta{}_\rho\right)p_1^\sigma p_2^\rho(p_4)_\eta(p_3)_\lambda\right]\\
&= \frac{64g^4}{s^2}(p_1\cdot p_4)(p_2\cdot p_3) = \frac{16g^4}{s^2}\left(u-m_e^2\right)^2
\end{aligned} \tag{10}$$

while with similar steps we get the analogous result for the matrix element, Eq. (9)

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{64g^4}{u^2}(p_1\cdot p_2)(p_3\cdot p_4) = \frac{16g^4}{u^2}\left(s-m_e^2\right)^2. \tag{11}$$

4. First of all,in order to evaluate the phase space in the laboratory reference frame, we write

down the kinematics for this $2 \to 2$ process

$$p_1 = (m_e, 0, 0, 0) \tag{12}$$
$$p_2 = (E_\nu, 0, 0, E_\nu) \tag{13}$$
$$p_3 = (E_e, \vec{p}_e) \tag{14}$$
$$p_4 = (p_4, \vec{p}_4)\,. \tag{15}$$

In general, the phase space is given by

$$d\Phi = \frac{d^3\vec{p}_3}{(2\pi)^3\, 2E_e} \frac{d^3\vec{p}_4}{(2\pi)^3\, 2p_4} (2\pi)^4\, \delta\left(E_\nu + m_e - E_e - p_4\right) \delta^{(3)}\left(\vec{p} - \vec{p}_e - \vec{p}_4\right), \tag{16}$$

which becomes in our case, by performing the integration over $\vec{p}_4$ with the three-dimensional delta constraint

$$d\Phi = \frac{p_e^2 dp_e d\cos\theta}{8\pi E_e p_4} \delta\left(E_\nu + m_e - E_e - p_4\right) \tag{17}$$

with $\vec{p} = E_\nu \hat{z}$ and

$$p_4 = |\vec{p} - \vec{p}_e| = \sqrt{E_\nu^2 + p_e^2 - 2p_e E_\nu \cos\theta}. \tag{18}$$

Now, using the hints, we further simplify this expression by integrating over $\cos\theta$ using the energy conservation delta, and by changing variables from $|\vec{p}_e|$ to the energy E_e. We get

$$\begin{aligned} d\Phi &= \frac{p_e^2 dp_e d\cos\theta}{8\pi E_e p_4} \frac{p_4}{p_e E_\nu} \delta\left(\cos\theta - \cos\theta_0\right) \\ &= \frac{p_e dp_e}{8\pi E_e E_\nu} = \frac{dE_e}{8\pi E_\nu}, \end{aligned} \tag{19}$$

where $\cos\theta_0$ is the solution to the equation $p_4 = E_\nu + m_e - E_e$ with p_4 given by Eq. (18), and the result does not depend on it.

Finally, the flux factor is

$$\Phi_0 = 4\sqrt{(p_1 \cdot p_2)^2} = 4m_e E_\nu. \tag{20}$$

5. We are now ready to express the differential cross section for the two processes Eq. (6) and Eq. (8) in terms of y. We get

$$\left.\frac{d\sigma}{dy}\right|_{e\bar{\nu}\to e\bar{\nu}} = \frac{g^4}{2\pi m_e E_\nu} \frac{(y_m + 1 - y)^2}{\left(1 + \frac{y_m}{2}\right)^2}, \tag{21}$$
$$\left.\frac{d\sigma}{dy}\right|_{e\nu\to e\nu} = \frac{g^4}{2\pi m_e E_\nu} \frac{1}{\left(\frac{y_m}{2} + 1 - y\right)^2}, \tag{22}$$

where we also defined $y_m = \frac{m_e}{E_\nu}$.

6. In the limit $m_e \to 0$, we can rewrite the Lagrangian in terms of the field doublet $\psi = \begin{pmatrix} e \\ \nu \end{pmatrix}$ as

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \begin{pmatrix} i\partial\!\!\!/ & \gamma^\mu\left(1 - \gamma_5\right) \\ \gamma^\mu\left(1 - \gamma_5\right) & i\partial\!\!\!/ \end{pmatrix} \psi. \tag{23}$$

Using the known expression of the Pauli matrices, we can rewrite Eq. (23) as

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ I + \gamma^\mu\left(1 - \gamma_5\right)\sigma_1\right)\psi, \tag{24}$$

where I and σ_1 are respectively the identity and the first Pauli matrix. It is now apparent that the Lagrangian Eq. (24) is invariant under the transformation

$$\psi = e^{i\theta I}\psi \tag{25}$$

and the transformation

$$\psi = e^{i\theta' \sigma_1}\psi, \tag{26}$$

which corresponds to a global $U(1) \times U(1)$ symmetry.