QUANTUM FIELD THEORY I

Solution

September 20, 2019

Consider a theory with two massless Dirac fermions given by the following Lagrangian:

$$\mathcal{L} = i\bar{\psi}_1 \partial\!\!\!/ \psi_1 + i\bar{\psi}_2 \partial\!\!\!/ \psi_2 + G\left(\bar{\psi}_1 \gamma^\mu \psi_2\right)\left(\bar{\psi}_2 \gamma^\mu \psi_1\right) \tag{1}$$

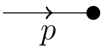
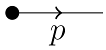
where the field ψ_i $(i = 1, 2)$ is defined as follows

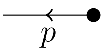
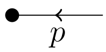
$$\psi_i(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_s \left\{ b_i^s(\vec{p}) u_i^s(p) \mathrm{e}^{-ip^\mu x_\mu} + c_i^{s^\dagger}(\vec{p}) v_i^s(p) \mathrm{e}^{ip^\mu x_\mu} \right\}. \tag{2}$$

The annihilation and creation operators satisfy the following anti-commutation relation:

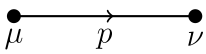
$$\{b_i^s(\vec{p}), b_i^{r^\dagger}(\vec{k})\} = \{c_i^s(\vec{p}), c_i^{r^\dagger}(\vec{k})\} = (2\pi)^3 \delta^{sr} \delta^{(3)}(\vec{p} - \vec{k}). \tag{3}$$

(1) Let us first write down the Feynman rules for this theory. The external fermion lines follow the following rules

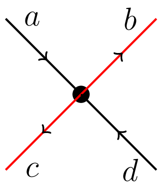
$\psi_i|f(p,s)\rangle =$  $= u_i^s(p)$ $\qquad$ $\langle f(p,s)|\bar{\psi}_i =$  $= \bar{u}_i^s(p)$

$\bar{\psi}_i|\bar{f}(p,s)\rangle =$  $= \bar{v}_i^s(p)$ $\qquad$ $\langle \bar{f}(p,s)|\psi_i =$  $= v_i^s(p)$,

while the fermion propagator and the 4-Fermi vertex are respectively given by



$$= \frac{i\not{p}_i}{p_i^2 + i\epsilon}, \tag{4}$$



$$= iG(\gamma^\mu)_{ac}(\gamma^\mu)_{bd}, \tag{5}$$

where the black line denotes fermion 1 and the red line fermion 2.

As one can extract from the Lagrangian, the real constant G has a dimension $[m]^{-2}$ and therefore the theory is not renormalizable.

(2) The Lagrangian is invariant under the following transformation:

$$\psi_i \longrightarrow \psi_i' = \mathrm{e}^{-iq_i}\psi_i. \tag{6}$$

The symmetry transformation form a $[U(1)]_{q_i}$ group which is associated to the conservation of the charge q_i of the fermion i. Since there are two conserved charges, the global symmetry group is therefore given by $[U(1)]_{q_1} \otimes [U(1)]_{q_2}$.

The Lagrangian is also invariant under the global chiral transformation:

$$\psi_i \longrightarrow \psi_i^{'} = \mathrm{e}^{-iq\gamma_5}\psi_i. \tag{7}$$

Note that this only holds if *both* the ψ_i are transformed at once.

Hence the overall symmetry is $\left[U_V(1)\right]_{q_1} \otimes \left[U_V(1)\right]_{q_2} \otimes \left[U_A(1)\right]$.

The Noether current corresponding to the vector symmetries is given by the following definition:

$$\mathcal{J}_i^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\Delta\psi_i, \qquad \text{where} \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)} = i\bar{\psi}_i\gamma^\mu. \tag{8}$$

The expression of $\Delta\psi_i$ can be derived from the infinitesimal transformation,

$$\psi_i \longrightarrow \psi_i + \Delta\psi_i \Longleftrightarrow \Delta\psi_i = -iq_i\psi_i \tag{9}$$

Hence, the expression of the current is given by

$$\mathcal{J}_i^\mu = q_i\bar{\psi}_i\gamma^\mu\psi_i. \tag{10}$$

For the axial symmetry one finds instead

$$\mathcal{J_A}^\mu = \sum_{1=1}^{2}\bar{\psi}_i\gamma^\mu\gamma_5\psi_i. \tag{11}$$

Let us now compute the expression of the corresponding conserved charges and express the final result in terms of the creation and annihilation operators. For the vector symmetries the Noether charge is defined as

$$\mathcal{Q}_i = \int \mathrm{d}^3x\, \mathcal{J}_i^0(x) = q_i \int \mathrm{d}^3x\, \psi_i^\dagger(x)\psi_i(x) \tag{12}$$

Expanding the above expression leads us to the following results

$$\begin{aligned}
\mathcal{Q}_i &= q_i \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{r,s} \left\{ b_i^{r\dagger}(\vec{p})b_i^s(\vec{p})u^{r\dagger}(p)u^s(p) + c_i^r(-\vec{p})c_i^{s\dagger}(-\vec{p})v^{r\dagger}(-p)v^s(-p) \right\} \\
&= q_i \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_s \left(b_i^{s\dagger}(\vec{p})b_i^s(\vec{p}) + c_i^s(-\vec{p})c_i^{s\dagger}(-\vec{p}) \right) && (13) \\
&= q_i \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_s \left(b_i^{s\dagger}(\vec{p})b_i^s(\vec{p}) - c_i^{s\dagger}(\vec{p})c_i^s(\vec{p}) \right). && (14)
\end{aligned}$$

In order to get the first line, we integrated over x then over the second momentum. From the first to the second line, we used the fact that $u^{r\dagger}(p)u^s(p) = v^{r\dagger}(p)v^s(p) = 2E_p\delta^{rs}$. Notice that by symmetry $c_i^s(-\vec{p})c_i^{s\dagger}(-\vec{p}) = c_i^s(\vec{p})c_i^{s\dagger}(\vec{p})$. To get to the last line, we wrote $c_i^s(\vec{p})c_i^{s\dagger}(\vec{p})$ in terms if its anti-commutation.

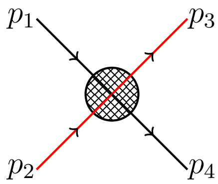
For the axial symmetry we analogously get

$$\mathcal{Q}_5 = \sum_i \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_s s_i \left(b_i^{s\dagger}(\vec{p})b_i^s(\vec{p}) + c_i^{s\dagger}(\vec{p})c_i^s(\vec{p}) \right), \tag{15}$$

where the eigenvalue of γ_5 is equal to s for particles and to $-s$ for antiparticles.

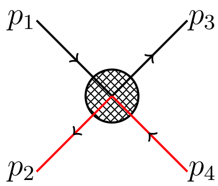
(3) Considering the following processes $f_1(p_1) + f_2(p_2) \to f_1(p_3) + f_2(p_4)$ and $f_1(p_1) + \bar{f}_2(p_2) \to f_1(p_3) + \bar{f}_2(p_4)$, one can draw the Feynman diagrams and use the Feynman rules to write down the corresponding expression of scattering amplitudes.

$$i\mathcal{M}_a(f_1 f_2 \to f_1 f_2) =$$



$$= iG\left[\bar{u_2}(p_3)\gamma^\mu u_1(p_1)\bar{u_1}(p_4)\gamma_\mu u_2(p_2)\right], \tag{16}$$

$$i\mathcal{M}_b(f_1 \bar{f}_2 \to f_1 \bar{f}_2) =$$



$$= iG\left[\bar{u_1}(p_3)\gamma^\mu v_2(p_4)\bar{v_2}(p_2)\gamma_\mu u_1(p_1)\right], \tag{17}$$

where again, as in Eq. (5), the black line denotes a fermion of type 1 and the red line a fermion of type 2, and the index on the spinors denote the species of fermion.

(4) We can now compute the square of the modulus of the unpolarized amplitudes given by the above Feynman diagrams.

First, let us consider the process $f_1(p_1) + f_2(p_2) \to f_1(p_3) + f_2(p_4)$. Multiplying Eq. (16) by its hermitian conjugate, summing over the polarization of the final state and averaging over the polarization of the initial states, we end up with the following result

$$|\bar{\mathcal{M}_a}|^2 = \frac{1}{4}\sum_{s_3,s_4} |\mathcal{M}_a|^2 = \frac{G^2}{4}\mathrm{Tr}\left(\not{p}_3\gamma^\mu\not{p}_1\gamma^\nu\right)\mathrm{Tr}\left(\not{p}_4\gamma^\mu\not{p}_2\gamma^\nu\right) \tag{18}$$

$$= 4G^2\left(p_3^\mu p_1^\nu - (p_3 p_1)g^{\mu\nu} + p_3^\nu p_1^\mu\right)\left(p_4^\mu p_2^\nu - (p_4 p_2)g^{\mu\nu} + p_4^\nu p_2^\mu\right) \tag{19}$$

$$= 8G^2\left((p_1 p_2)(p_3 p_4) + (p_1 p_4)(p_2 p_3)\right). \tag{20}$$

Taking a similar approach, Eq. (17) yields

$$|\bar{\mathcal{M}_b}|^2 = \frac{1}{4}\sum_{s_3,s_4} |\mathcal{M}_b|^2 = \frac{G^2}{4}\mathrm{Tr}\left(\not{p}_3\gamma^\mu\not{p}_4\gamma^\nu\right)\mathrm{Tr}\left(\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\right) \tag{21}$$

$$= 4G^2\left(p_3^\mu p_4^\nu - (p_3 p_4)g^{\mu\nu} + p_3^\nu p_4^\mu\right)\left(p_2^\mu p_1^\nu - (p_2 p_1)g^{\mu\nu} + p_2^\nu p_1^\mu\right) \tag{22}$$

$$= 8G^2\left((p_1 p_4)(p_2 p_3) + (p_1 p_3)(p_2 p_4)\right). \tag{23}$$

Expressing the results in terms of the Mandelstam variables (Eq. (24)), one can clearly see that the two scattering amplitudes are related by crossing symmetry, by a swap of s and u.

$$|\bar{\mathcal{M}_a}|^2 = 2G^2\left(s^2 + u^2\right), \qquad |\bar{\mathcal{M}_b}|^2 = 2G^2\left(u^2 + t^2\right). \tag{24}$$

(5) Let us add an extra-contribution to the Lagrangian,

$$\Delta\mathcal{L} = GT^\mu T^{\dagger\mu}, \tag{25}$$

where T^μ is of the form

$$T^\mu \equiv \lambda_1 \bar{\psi}_1 \gamma^\mu \psi_1 + \lambda_2 \bar{\psi}_2 \gamma^\mu \psi_2. \tag{26}$$

Let us then determine the values of the parameters λ_i such that the new Lagrangian can be written in the following form

$$\mathcal{L} = i\bar{\psi}_1 \partial\!\!\!/ \psi_1 + i\bar{\psi}_2 \partial\!\!\!/ \psi_2 + \frac{G}{4} \sum_{i,a,b,a',b'} \left(\bar{\psi}_a \gamma^\mu \sigma^i_{ab} \psi_b\right) \left(\bar{\psi}_{a'} \gamma^\mu \sigma^i_{a'b'} \psi_{b'}\right). \tag{27}$$

First, expanding Eq. (25) yields the following expression:

$$\begin{aligned} \Delta\mathcal{L} = G \Big\{ & \lambda_1 \lambda_1^\dagger (\bar{\psi}_1 \gamma^\mu \psi_1)(\bar{\psi}_1 \gamma^\mu \psi_1) + \lambda_2 \lambda_2^\dagger (\bar{\psi}_2 \gamma^\mu \psi_2)(\bar{\psi}_2 \gamma^\mu \psi_2) \\ & + \lambda_1 \lambda_2^\dagger (\bar{\psi}_1 \gamma^\mu \psi_1)(\bar{\psi}_2 \gamma^\mu \psi_2) + \lambda_1 \lambda_2^\dagger (\bar{\psi}_1 \gamma^\mu \psi_1)(\bar{\psi}_2 \gamma^\mu \psi_2) \Big\}. \end{aligned} \tag{28}$$

Comparing the terms in the above equation to the expanded version of Eq. (27), one finds the following system of equations:

$$\begin{cases} \lambda_1 \lambda_1^\dagger = \lambda_2 \lambda_2^\dagger = 1/4, \\ \lambda_1 \lambda_2^\dagger = \lambda_2 \lambda_1^\dagger = -1/4. \end{cases} \tag{29}$$

By solving this equation, we find that:

$$\lambda_1 = \frac{\mathrm{e}^{i\alpha}}{2}, \qquad \lambda_2 = -\frac{\mathrm{e}^{i\alpha}}{2}. \tag{30}$$

(6) In addition to the $[U(1)]_{q_1} \otimes [U(1)]_{q_2}$ symmetry, the new Lagrangian in Eq. (27) possesses a more general symmetry given by the $SU(2)$ group. Indeed, Eq. (27) can be written in terms of a doublet $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ in the following way

$$\mathcal{L} = i\bar{\Psi} \partial\!\!\!/ I \Psi + \tilde{G} \sum_i \left(\bar{\Psi} \gamma^\mu \sigma^i \Psi\right) \left(\bar{\Psi} \gamma^\mu \sigma^i \Psi\right) \quad \text{with} \quad \tilde{G} = G/4, \tag{31}$$

and one can notice that this expression is invariant under the transformation:

$$\Psi \longrightarrow \Psi' = \mathrm{e}^{i\theta_i \sigma_i} \Psi. \tag{32}$$

The corresponding Noether current is then given by $\mathcal{J}^a_\mu = \Psi \gamma_\mu \sigma^a \Psi$.

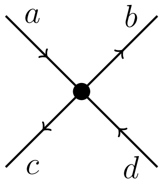
(7) Let us now compute the commutator of the Noether charges derived from question (6). Using the definition of the canonical (anti)commutation relations $\{\pi(y), \Psi(x)\} = -i\delta^{(3)}(x - y)$ and the commutation relation defined by the Pauli matrices $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$, it follows that

$$[\mathcal{Q}_a, \mathcal{Q}_b] = 2i\varepsilon_{abc} \mathcal{Q}_c, \tag{33}$$

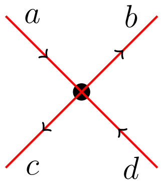
where as usual, the charge $\mathcal{Q}$ is defined as:

$$\mathcal{Q}_a = \int \mathrm{d}^3x\, \mathcal{J}^0_a = \int \mathrm{d}^3x\, \Psi^\dagger \sigma_a \Psi. \tag{34}$$

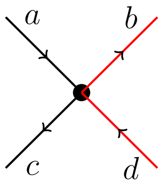
(8) There are now four new Feynman rules, corresponding to the four terms in Eq. (28):



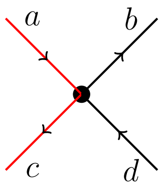
$$= iG(\gamma^\mu)_{ac}(\gamma^\mu)_{bd}. \tag{35}$$



$$= iG(\gamma^\mu)_{ac}(\gamma^\mu)_{bd}. \tag{36}$$



$$= iG(\gamma^\mu)_{ac}(\gamma^\mu)_{bd}. \tag{37}$$



$$= iG(\gamma^\mu)_{ac}(\gamma^\mu)_{bd}. \tag{38}$$

These differ from the previous rule because now a fermion-antifermion pair of either type can annihilate and then produce a fermion-antifermion pair again of either type, thus giving the four listed combinations, while the vertex Eq. (5) only allowed annihilation of a fermion of type 1 and an antifermion of type 2, with creation of a fermion of type 2 and an antifermion of type 1, and all the crossings obtained from this by turning an incoming fermion into an outgoing antifermion and conversely.