QUANTUM FIELD THEORY I

Solution

September 22nd, 2021

Consider a massless theory given by the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\sum_{i=1,2}\bar{\psi}_i \partial\!\!\!/ \psi_i + \sum_{i=1,2}\bar{\psi}_i A\!\!\!/ \left(a + b\gamma_5\right)\psi_i, \tag{1}$$

where $F_{\mu\nu}$ is the Maxwell field strength and ψ is the Dirac fermion field.

- The Maxwell field tensor is defined as $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$, where the expression of the photon field is given by

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sum_\lambda \left(\epsilon_{\mu,\lambda}(p)a_{\vec{p},\lambda}\,\mathrm{e}^{-ip^\mu x_\mu} + \epsilon^*_{\mu,\lambda}(p)a^\dagger_{\vec{p}}\,\mathrm{e}^{ip^\mu x_\mu}\right). \tag{2}$$

- The Dirac fermion field ψ_i is defined as follows

$$\psi_i(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sum_{s_i}\left(b^{s_i}_{\vec{p}}\,u^{s_i}_i(p)\mathrm{e}^{-ip^\mu x_\mu} + c^{s_i\dagger}_{\vec{p}}v^{s_i}_i(p)\mathrm{e}^{ip^\mu x_\mu}\right), \tag{3}$$

  with the annihilation and creation operators satisfying the following anticommutation relation:

$$\{b^s_{\vec{p}}, b^{r\dagger}_{\vec{k}}\} = \{c^s_{\vec{p}}, c^{r\dagger}_{\vec{k}}\} = (2\pi)^3\delta^{sr}\delta^{(3)}(\vec{p}-\vec{k}). \tag{4}$$

(1) Let us first write down the Feynman rules of the theory.

(a) External lines:

$$\psi_i|f_i(p,s)\rangle \equiv \underset{p}{\longrightarrow}\bullet = u^{s_i}_i(p), \qquad \langle f_i(p,s)|\bar{\psi}_i \equiv \bullet\underset{p}{\longrightarrow} = \bar{u}^{s_i}_i(p),$$

$$\bar{\psi}_i|\bar{f}_i(p,s)\rangle \equiv \underset{p}{\longleftarrow}\bullet = \bar{v}^{s_i}_i(p), \qquad \langle \bar{f}_i(p,s)|\psi_i \equiv \bullet\underset{p}{\longleftarrow} = v^{s_i}_i(p),$$

$$A_\mu|\gamma(p,s)\rangle \equiv \underset{p}{\sim\!\sim\!\sim}\bullet = \epsilon_\mu(p,s), \qquad \langle\gamma(p,s)|A_\mu \equiv \bullet\underset{p}{\sim\!\sim\!\sim} = \epsilon^*_\mu(p,s). \tag{5}$$

(b) Propagators:

$$\text{Fermion}: \quad \bullet\underset{p}{\longrightarrow}\bullet = \frac{i p\!\!\!/}{p^2 + i\epsilon} \tag{6}$$

$$\text{Photon}: \quad \underset{\mu}{\bullet}\underset{p}{\sim\!\sim\!\sim\!\sim}\underset{\nu}{\bullet} = -\frac{i}{p^2+i\epsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right] \tag{7}$$

In the expression of the photon propagator, ξ parametrizes a set of covariant gauges (for a Feynman gauge, $\xi = 1$).

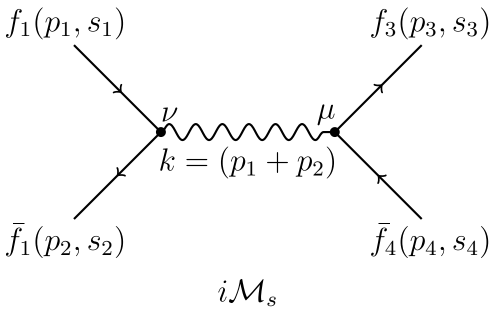
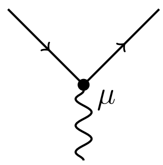


Figure 1: Leading non-vanishing diagrams for $f_1\bar{f}_1 \longrightarrow f_2\bar{f}_2$.

(c) Vertex:



$$= i\gamma^\mu \left(a + b\gamma_5\right)$$

(2) The energy-momentum tensor for this theory is given by:

$$T^\mu_{\ \nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\nu A_\lambda + \sum_{i=1,2} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)}\partial_\nu \psi_i - \delta^\mu_{\ \nu}\mathcal{L} \tag{8}$$

$$= -F^{\mu\lambda}\partial_\nu A_\lambda + i\sum_{i=1,2} \bar{\psi}_i\gamma^\mu\partial_\nu\psi_i - \delta^\mu_{\ \nu}\mathcal{L}. \tag{9}$$

The Hamiltonian density $\mathcal{H} = T^{00}$ is then given by:

$$\mathcal{H} = -F^{0\lambda}\dot{A}_\lambda + i\sum_{i=1,2} \psi_i^\dagger\dot{\psi}_i - g^{00}\mathcal{L} \tag{10}$$

$$= F_{\mu\nu}F^{\mu\nu} - F^{0\lambda}\dot{A}_\lambda + i\sum_{i=1,2}\bar{\psi}_i\vec{\gamma}\vec{\nabla}\psi_i - \sum_{i=1,2}\bar{\psi}_i \not{A}\left(a + b\gamma_5\right)\psi_i \tag{11}$$

$$= \frac{1}{2}\left(F^{ij}F^{ij} + F^{0i}F^{0i}\right) + i\sum_{i=1,2}\bar{\psi}_i\vec{\gamma}\vec{\nabla}\psi_i - \sum_{i=1,2}\bar{\psi}_i \not{A}\left(a + b\gamma_5\right)\psi_i, \tag{12}$$

where the last step holds assuming $A^0 = 0$

(3) Let us now compute the scattering amplitude of the process $f_1\bar{f}_1 \to f_2\bar{f}_2$. As shown in Fig.1, only one single diagram contribute to such a process at leading order. Using the Feynman rules presented above, the scattering amplitude $i\mathcal{M}_s\left(p_1, p_2; p_3, p_4\right)$– that for brevity we simply denote by $i\mathcal{M}_s$–can be written as:

$$i\mathcal{M}_s = \bar{u}_2^{s_3}(p_3)\left[\gamma^\mu\left(a + b\gamma_5\right)\right]v_2^{s_4}(p_4)\left(\frac{ig_{\mu\nu}}{k^2}\right)u_1^{s_1}(p_1)\left[\gamma^\nu\left(a + b\gamma_5\right)\right]\bar{v}_1^{s_2}(p_2) \tag{13}$$

$$= \left(\frac{i}{s}\right)\bar{u}_2^{s_3}(p_3)\left[\gamma^\mu\left(a + b\gamma_5\right)\right]v_2^{s_4}(p_4)u_1^{s_1}(p_1)\left[\gamma_\mu\left(a + b\gamma_5\right)\right]\bar{v}_1^{s_2}(p_2) \tag{14}$$

Taking the modulus square of the above amplitude leads to the following expression:

$$
\begin{aligned}
|\mathcal{M}_s|^2 = \frac{1}{s^2}&\mathrm{Tr}\left\{\bar{v}_4^{s_4}(p_4)\left[\gamma^\nu\left(a+b\gamma_5\right)\right]u_2^{s_3}(p_3)\bar{u}_2^{s_3}(p_3)\left[\gamma^\mu\left(a+b\gamma_5\right)\right]v_2^{s_4}(p_4)\right\}\\
&\times\mathrm{Tr}\left\{\bar{v}_1^{s_2}(p_2)\left[\gamma^\nu\left(a+b\gamma_5\right)\right]u_1^{s_1}(p_1)\bar{u}_1^{s_1}(p_1)\left[\gamma^\mu\left(a+b\gamma_5\right)\right]v_1^{s_2}(p_2)\right\}. \qquad (15)
\end{aligned}
$$

Since we are considering an unpolarized process, we have to average over the polarization of the initial state fermions and sum over the polarization of the final state fermions. This yields:

$$
\begin{aligned}
|\bar{\mathcal{M}}_s|^2 = \frac{1}{4s^2}&\mathrm{Tr}\left\{\not{p}_4\left[\gamma^\nu\left(a+b\gamma_5\right)\right]\not{p}_3\left[\gamma^\mu\left(a+b\gamma_5\right)\right]\right\}\\
&\times\mathrm{Tr}\left\{\not{p}_2\left[\gamma_\nu\left(a+b\gamma_5\right)\right]\not{p}_1\left[\gamma_\mu\left(a+b\gamma_5\right)\right]\right\}, \qquad (16)
\end{aligned}
$$

where to go from Eq. (15) to Eq. (16) we used the completeness relations which in the massless case are given by:

$$
\sum_s \bar{u}^s(p)u^s(p) = \sum_s \bar{v}^s(p)v^s(p) = \not{p}. \qquad (17)
$$

Each trace appearing in Eq. (16) can therefore be simplified as follows:

$$
\begin{aligned}
\mathrm{Tr}\left\{\not{p}_i\left[\gamma^\nu\left(a+b\gamma_5\right)\right]\not{p}_j\left[\gamma^\mu\left(a+b\gamma_5\right)\right]\right\} &= a^2\mathrm{Tr}\left(\not{p}_i\gamma^\nu\not{p}_j\gamma^\mu\right) \qquad (18)\\
&+2(ab)\mathrm{Tr}\left(\not{p}_i\gamma^\nu\not{p}_j\gamma^\mu\gamma_5\right)+b^2\mathrm{Tr}\left(\not{p}_i\gamma^\nu\gamma_5\not{p}_j\gamma^\mu\gamma_5\right)
\end{aligned}
$$

Using the properties of γ^5, specifically $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ and $(\gamma^5)^2 = \mathbb{I}$, and factoring out the momenta from the trace, we arrive at the following expression

$$
\begin{aligned}
\mathrm{Tr}\left\{\not{p}_i\left[\gamma^\nu\left(a+b\gamma_5\right)\right]\not{p}_j\left[\gamma^\mu\left(a+b\gamma_5\right)\right]\right\} &= \left(a^2+b^2\right)(p_i)_\alpha(p_j)_\beta\,\mathrm{Tr}\left(\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu\right) \qquad (19)\\
&+2(ab)\,(p_i)_\alpha(p_j)_\beta\,\mathrm{Tr}\left(\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu\gamma_5\right).
\end{aligned}
$$

The trace identities can now be applied to further simplify the above expressions. For a generic trace, doing so yields to the following expression

$$
\begin{aligned}
\mathrm{Tr}\left\{\not{p}_i\left[\gamma^\nu\left(a+b\gamma_5\right)\right]\not{p}_j\left[\gamma^\mu\left(a+b\gamma_5\right)\right]\right\} &= 4\left(a^2+b^2\right)\left(p_i^\mu p_j^\nu + p_i^\nu p_j^\mu - g^{\mu\nu}(p_ip_j)\right) \qquad (20)\\
&+8i(ab)\varepsilon^{\alpha\nu\beta\mu}(p_i)_\alpha(p_j)_\beta,
\end{aligned}
$$

where $\varepsilon^{\alpha\nu\beta\mu}$ is the Levi-Civita tensor and (p_ip_j) represents the standard dot product between the two momenta p_i and p_j. It follows from the above expression that Eq. (16) can be written as

$$
\begin{aligned}
|\bar{\mathcal{M}}_s|^2 = \frac{4}{s^2}&\left[\left(a^2+b^2\right)\left(p_4^\mu p_3^\nu + p_4^\nu p_3^\mu - g^{\mu\nu}(p_3p_4)\right)+2i(ab)\varepsilon^{\alpha\nu\beta\mu}(p_4)_\alpha(p_3)_\beta\right] \qquad (21)\\
&\times\left[\left(a^2+b^2\right)\left((p_2)_\mu(p_1)_\nu+(p_2)_\nu(p_1)_\mu - g_{\mu\nu}(p_3p_4)\right)+2i(ab)\varepsilon_{\sigma\nu\rho\mu}p_2^\sigma p_1^\rho\right]
\end{aligned}
$$

Expanding the above expression and performing some algebraic simplifications, we arrive to the following expression:

$$
|\bar{\mathcal{M}}_s|^2 = \frac{8}{s^2}\left[\left(a^2+b^2\right)^2\left((p_1p_3)(p_2p_4)+(p_1p_4)(p_2p_3)\right)-2\varepsilon^{\alpha\nu\beta\mu}\varepsilon_{\sigma\nu\rho\mu}(p_4)_\alpha(p_3)_\beta\,p_2^\sigma p_1^\rho\right] \qquad (22)
$$

We can get rid of the Levi-Civita tensors by recalling that:

$$\varepsilon^{\alpha\nu\beta\mu}\varepsilon_{\sigma\nu\rho\mu} = -2\left(g^{\alpha}_{\sigma}g^{\beta}_{\rho} - g^{\alpha}_{\rho}g^{\beta}_{\sigma}\right). \tag{23}$$

Plugging Eq. (23) into Eq. (22) and performing some simplifications, one can show that the final expression of the amplitude square is given by the following:

$$|\bar{\mathcal{M}}_s|^2 = \frac{8}{s^2}\left[A(a,b)(p_1p_3)(p_2p_4) + B(a,b)(p_1p_4)(p_2p_3)\right]. \tag{24}$$

were we defined A and B as:

$$A(a,b) = (a^2+b^2)^2 + 4(ab)^2, \qquad B(a,b) = \left(a^2-b^2\right)^2. \tag{25}$$

(4) Let us now express the above amplitude in terms of the Mandelstam variables. Recall that in the massless case, the Mandelstam variables are defined as :

$$s = 2(p_1p_2) = 2(p_3p_4), \quad t = 2(p_1p_3) = 2(p_2p_4), \quad u = 2(p_1p_4) = 2(p_2p_3). \tag{26}$$

Using these definitions, it is straightforward to see that Eq. (24) becomes

$$|\bar{\mathcal{M}}_s|^2 = \frac{64}{s^2}\left[A(a,b)\,t^2 + B(a,b)\,u^2\right]. \tag{27}$$

(5) By inspecting Eq. (24), it is clear that upon swapping the momenta in the final states A and B are interchanged. Hence, the amplitude is invariant only if $A = B$ which in turn holds only if $a = 0$ or $b = 0$.

The forward-backward asymmetry is given by

$$A_{\mathrm{FB}} = \frac{|\bar{\mathcal{M}}_s\left(p_1,p_2;p_1,p_2\right)|^2}{|\bar{\mathcal{M}}_s\left(p_1,p_2;p_2,p_1\right)|^2}, \tag{28}$$

in which the set of momenta appearing in the initial states are the same as the ones appearing in the final states. Using Eq. (24), we see that:

$$A_{\mathrm{FB}} = \frac{(a^2-b^2)^2}{(a^2+b^2)^2+4(ab)^2}. \tag{29}$$

Eq. (29) shows that for a generic values of a and b where $a \neq \pm b$, there is an asymmetry between the two amplitudes.

This vanishes when $a = \pm b$, A_{FB} and it is equal to one when $a = 0$ or $b = 0$. The two cases in which $a = 0$ or $b = 0$ correspond to having only vector or only axial vector couplings. The cases in which $a = \pm b$ correspond to having only $V + A$ or only $V - A$ couplings, i.e. to having only couplings that involve the ψ_+ or the ψ_- field combination (see the next question).

(6) Let us finally express the Lagrangian given in Eq. (1) in terms of $\psi_i^{\pm} = P_{\pm}\psi_i$ where the projector is defined as $P_{\pm} = (1 \pm \gamma_5)/2$. Doing so leads to the following expression

$$\begin{aligned}\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\sum_{i=1,2}\bar{\psi}_i^{+}\partial\!\!\!/\psi_i^{+} + i\sum_{i=1,2}\bar{\psi}_i^{-}\partial\!\!\!/\psi_i^{-} \\ + (a+b)\sum_{i=1,2}\bar{\psi}_i^{+}A\!\!\!/\psi_i^{+} + (a-b)\sum_{i=1,2}\bar{\psi}_i^{-}A\!\!\!/\psi_i^{-}.\end{aligned} \tag{30}$$

This means that the Lagrangian splits into the sum of a Lagrangian for the ψ_- field only and a Lagrangian for the ψ_+ field only. These two field have different couplings to the gauge field, equal to $a+b$ and $a-b$ respectively. The cases considered in the previous problem are seen to correspond to the couplings of the $\psi_\pm$ fields being the same in magnitude ($a=0$ or $b=0$) or to one of the two couplings being zero ($a=b$ or $a=-b$).

If we put together the fields ψ_1 and ψ_2 in a two-component complex vector

$$\psi^\pm = \begin{pmatrix} \psi_1^\pm \\ \psi_2^\pm \end{pmatrix} \tag{31}$$

then the Lagrangian can be rewritten as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}^+ \not{\partial}\psi_i^+ + (a+b)\,\bar{\psi}^+ \not{A}\psi^+ + (a-b)\,\bar{\psi}^- \not{A}\psi^-.$$

It is clear that this is invariant upon any transformation of the form

$$\psi^\pm \longrightarrow U^\pm \psi^\pm \tag{32}$$

where $U^\pm$ are two independent 2x2 unitary matrices that transform the fields ψ_i^+ and ψ^- respectively. A 2x2 unitary matrix can be decomposed into the product of a unitary matrix with determinant equal to 1 (SU(2)) times a phase transformation (U(1)). The overall symmetry is thus U(2)xU(2)=SU(2)xU(1)xSU(2)xU(1).

The Noether currents are

$$\mathcal{J}^{\mu,\pm} = \bar{\psi}^\pm \gamma^\mu \psi^\pm; \qquad \mathcal{J}_a^{\mu,\pm} = \bar{\psi}^\pm \sigma_a \gamma^\mu \psi^\pm, \tag{33}$$

where σ_a are Pauli matrices acting on the two-component complex vectors Eq. (31).