Solution of the exam of Theoretical Physics of September 2023

September 13, 2023

Real scalar field:

$$\phi_i(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{p,i} e^{-ipx} + a^\dagger_{p_i} e^{ipx} \right) . \tag{1}$$

1. The energy-momentum tensor is defined as

$$T^\mu{}_\nu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i - \delta^\mu{}_\nu \mathcal{L} \, . \tag{2}$$

Using

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = \partial^\mu \phi_i \, , \tag{3}$$

we obtain

$$T^\mu{}_\nu = \sum_i \partial^\mu \phi_i \partial_\nu \phi_i - \delta^\mu{}_\nu \mathcal{L} \, . \tag{4}$$

The hamiltonian density is

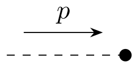
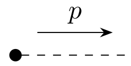
$$\begin{aligned} \mathcal{H} = T^{00} &= \sum_j \left(\partial^0 \phi_j \partial^0 \phi_j - \frac{1}{2} \partial_\mu \phi_j \partial^\mu \phi_j + \frac{1}{2} m_j^2 \phi_j^2 \right) - g \phi_1 \phi_2 \phi_3 \\ &= \sum_j \left(\partial^0 \phi_j \partial^0 \phi_j - \frac{1}{2} \partial^0 \phi_j \partial^0 \phi_j - \frac{1}{2} \partial_i \phi_j \partial^i \phi_j + \frac{1}{2} m_j^2 \phi_j^2 \right) - g \phi_1 \phi_2 \phi_3 \\ &= \frac{1}{2} \sum_j \left(\dot{\phi}_j \dot{\phi}_j + \vec{\nabla} \phi_j \cdot \vec{\nabla} \phi_j + m_j^2 \phi_j^2 \right) - g \phi_1 \phi_2 \phi_3 . \end{aligned} \tag{5}$$

2. A theory is renormalizable or not according to the dimensionality of the coupling. In order to determine the dimension of g we start from the fact that the action is dimensionless and therefore the Lagrangian density has dimension $[\mathcal{L}] = [M]^4$ where M is a mass unit. Remembering that $[\phi] = [M]$ we find

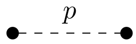
$$[g] = [M] \, , \tag{6}$$

therefore the theory is super-renormalizable.

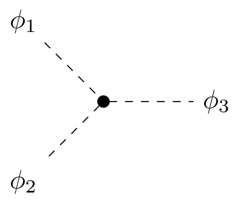
3.
- **External lines**

$\phi \, |s(p)\rangle =$  $= 1 \, ,$ $\qquad \langle s(p)| \, \phi =$  $= 1 \, .$ (7)

- **Propagators**

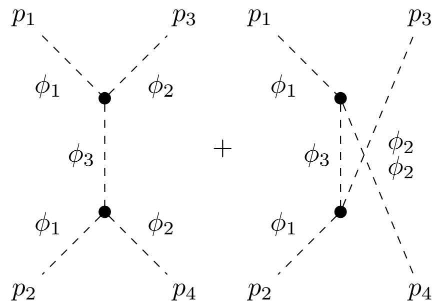
 $= \dfrac{i}{p^2 - m_i^2 + i\epsilon} \, .$ (8)

- **Vertex**

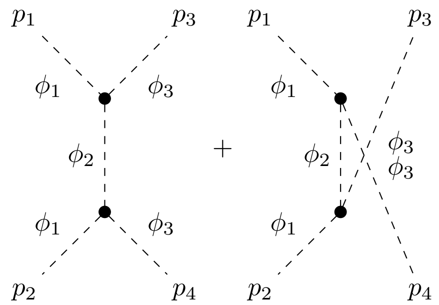
 $= ig \, .$ (9)

4. The numbers of loops L, of vertices V and external lines E in a theory with cubic coupling are related by $L = 1 + \frac{1}{2}(V - E)$. The given diagrams have $E = 4$ and therefore at tree level with $L = 0$ one gets $V = 2$, i.e. only diagrams with two vertices contribute to this process. Because in the given theory all lines entering a vertex are different, if the initial-state particles are the same then also the final-state particles must be the same and the process is mediated by either t channel or u channel excange of the third remaining particle. In other words, the diagrams are $\phi_1\phi_1 \to \phi_2\phi_2$ with t or u channel ϕ_3 exchange, or the same but with ϕ_2 and ϕ_3 interchanged.

5. The diagrams described in the previous point are



$$= i\mathcal{M}_t^{(3)} + i\mathcal{M}_u^{(3)} = i\mathcal{M}^{(3)}\,, \tag{10}$$



$$= i\mathcal{M}_t^{(2)} + i\mathcal{M}_u^{(2)} = i\mathcal{M}^{(2)}\,, \tag{11}$$

where label 2 or 3 refers to the field in the propagator.

6. The amplitudes $i\mathcal{M}^{(i)}$ are

$$i\mathcal{M}^{(i)} = i\mathcal{M}_t^{(i)} + i\mathcal{M}_u^{(i)} = (ig)^2 \frac{i}{(p_1 - p_3)^2 - m_i^2} + (ig)^2 \frac{i}{(p_1 - p_4)^2 - m_i^2}\,. \tag{12}$$

The squared amplitudes are just the trivial square modulus of the r.h.s. of Eq. (12), given that all particles are scalars.

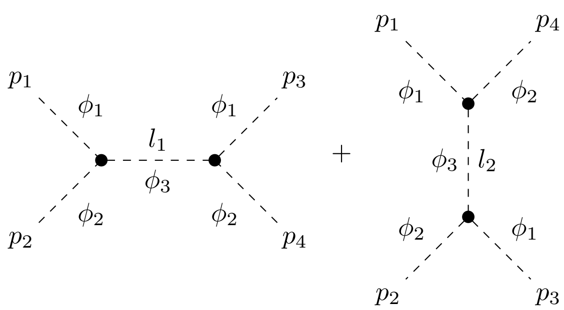
7. Defining the Mandelstam variables as

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2\,, \tag{13}$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2\,, \tag{14}$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2\,, \tag{15}$$

we have

$$i\mathcal{M}^{(i)} = -g^2 \left(\frac{i}{t - m_i^2} + \frac{i}{u - m_i^2} \right)\,. \tag{16}$$

8. By the same reasoning as in point (4), now the final-state particles must be the same as the initial-state ones and the nonvanishing amplitudes are those obtained from the Feynman diagrams



$$= i\mathcal{M}_s + i\mathcal{M}_u = i\mathcal{M}\,, \tag{17}$$

It immediately follows that

$$i\mathcal{M}_s = (ig)^2 \frac{i}{(p_1+p_2)^2 - m_3^2} = -ig^2 \frac{i}{s-m_3^2}\,, \tag{18}$$

$$i\mathcal{M}_u = (ig)^2 \frac{i}{(p_4-p_1)^2 - m_3^2} = -ig^2 \frac{i}{u-m_3^2} \tag{19}$$

9. It is clear that the diagrams in Eq. (10) can be related to the ones in Eq. (17) by

$$i\mathcal{M}\left\{\phi_1(p_1)+\phi_2(p_2)\to\phi_1(p_3)+\phi_2(p_4)\right\} = i\mathcal{M}\left\{\phi_1(p_1)+\phi_1(-p_3)\to\phi_2(-p_2)+\phi_2(p_4)\right\}\,. \tag{20}$$

In terms of momenta this is seen to correspond to

$$p_2 \to -p_3\,, \quad p_3 \to -p_2, \tag{21}$$

and in terms of Mandelstam invariants to

$$s \to t\,, \quad u \to u\,. \tag{22}$$

10. The amplitude is nonzero in the region in which it is possible to satisfy the momentum conservation Dirac delta that fixes $p_1+p_2 = p_3+p_4$. For the process of point (8), the initial-state and final-state particles are the same, and therefore this can always be satisfied for any value of the incoming particle momenta, so, choosing for example the center-of-mass frame in which the three momenta of the incoming particles are equal and opposite, $\vec{p}_3 = -\vec{p}_4$, for any value

$$p \geq 0; \quad E_i \geq m_i \tag{23}$$

of $p = |\vec{p}|$ and the energies E_1, E_2 of the incoming particles.

11. Because these are tree-level amplitudes, they can only acquire an imaginary part if some propagator has a vanishing denominator so that the $i\epsilon$ prescription is effective. In question 6 the denominators are never zero, so the amplitudes never get an imaginary part. This can only happen for the s-channel diagram of point (8), whenever $s = m_3^2$, i.e. when the energy of the incoming ϕ_1 and ϕ_2 particles is sufficiently high to produce an on-shell ϕ_3 particle.