Solutions to the exam of QFT1 of 18 September 2025

$$\mathcal{L} = \sum_{i=1}^{3} \frac{1}{2}(\partial_\mu \phi_i \partial^\mu \phi_i - M^2 \phi_i^2) + \sum_{a=1}^{2} \bar{\psi}_a (i\partial\!\!\!/ - m)\psi_a + \mathcal{L}_i \tag{1}$$

$$\mathcal{L}_i = -g \sum_{i=1}^{3} \left(\sum_{a,b=1}^{2} \bar{\psi}_a \sigma^i_{ab} \psi_b \right) \phi_i \tag{2}$$

$$= -g \sum_{i=1}^{3} \left(\bar{\Psi} \sigma^i \Psi \right) \phi_i, \tag{3}$$

where the last equality is when writing in terms of the spinor doublet $\Psi = (\psi_1, \psi_2)$.

(1) The energy-momentum tensor is

$$T^{\mu\nu} = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial^\nu \phi_i + \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)} \partial^\nu \psi_a - \eta^{\mu\nu} \mathcal{L} \tag{4}$$

$$= \sum_i \partial^\mu \phi_i \partial^\nu \phi_i + \sum_a \partial^\mu \bar{\psi}_a i \gamma^\mu \partial^\nu \psi_a - \eta^{\mu\nu} \mathcal{L}. \tag{5}$$

The Hamiltonian density is the 00'th component of this, and it is given by

$$\begin{aligned}\mathcal{H} = T^{00} = &\sum_i \dot{\phi}_i^2 + \sum_a \bar{\psi}_a i \gamma^0 \dot{\psi}_a - \sum_i \frac{1}{2} \left(\dot{\phi}_i^2 - (\nabla \phi_i)^2 - M^2 \phi_i^2 \right) \\ &- \sum_a \left(\bar{\psi}_a i \gamma^0 \dot{\psi}_a + \bar{\psi}_a i \vec{\gamma} \cdot \vec{\nabla} \psi_a - \bar{\psi}_a m \psi_a \right) + g \sum_i \left(\sum_a \bar{\psi}_a \sigma^i_{ab} \psi_b \right) \phi_i \end{aligned} \tag{6}$$

$$= \sum_i \left(\frac{1}{2} \dot{\phi}_i^2 + \frac{1}{2} (\nabla \phi_i)^2 + \frac{1}{2} M^2 \phi_i^2 \right) + \sum_a \left(-\bar{\psi}_a i \vec{\gamma} \cdot \vec{\nabla} \psi_a + \bar{\psi}_a m \psi_a \right) + g \sum_i \left(\sum_a \bar{\psi}_a \sigma^i_{ab} \psi_b \right) \phi_i. \tag{7}$$

(2) In terms of $\phi^\pm$, we can write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_3 \partial^\mu \phi_3 - M^2 \phi_3^2) + \partial_\mu (\phi^+)^* \partial^\mu \phi^+ - M^2 |\phi^+|^2 + \sum_a \bar{\psi}_a (i\partial\!\!\!/ - m)\psi_a + \mathcal{L}_i \tag{8}$$

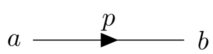
$$= \frac{1}{2}(\partial_\mu \phi_3 \partial^\mu \phi_3 - M^2 \phi_3^2) + \partial_\mu \phi^+ \partial^\mu \phi^- - M^2 \phi^+ \phi^- + \sum_a \bar{\psi}_a (i\partial\!\!\!/ - m)\psi_a + \mathcal{L}_i. \tag{9}$$

Introducing $\sigma_{\pm} = \frac{\sigma_1 \pm i\sigma_2}{2}$, the interaction Lagrangian in terms of $\phi_{\pm}$ can be written as

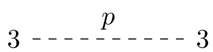
$$\mathcal{L}_i = -g\left(\bar{\Psi}\sigma^3\Psi\phi_3 + \sqrt{2}\bar{\Psi}\sigma_-\Psi\phi_- + \sqrt{2}\bar{\Psi}\sigma_+\Psi\phi_+\right) \tag{10}$$

$$= -g\left(\bar{\Psi}\sigma^3\Psi\phi_3 + \sqrt{2}\bar{\psi}_2\psi_1\phi^- + \sqrt{2}\bar{\psi}_1\psi_2\phi^+\right) \tag{11}$$

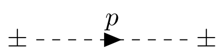
(3) The Feynman rules corresponding to this theory are the following: the propagators are given by



$= \frac{i(\not{p}+m)}{p^2-M^2+i\epsilon}\delta_{ab},\ a, b \in 1, 2$

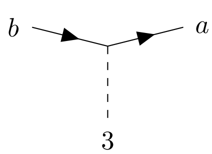


$= \frac{i}{p^2 - M^2 + i\epsilon}$

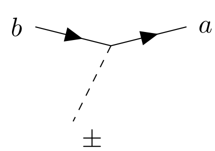


$= \frac{i}{p^2 - M^2 + i\epsilon}$

The vertices are

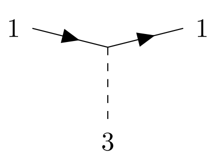


$= -ig\sigma^3_{ab}$

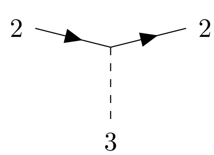


$= -\sqrt{2}ig\sigma^{\pm}_{ab}$

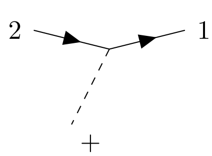
The vertices can be written without the Pauli matrices as



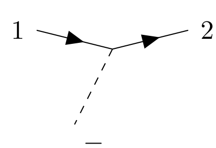
$= -ig$



$= +ig$

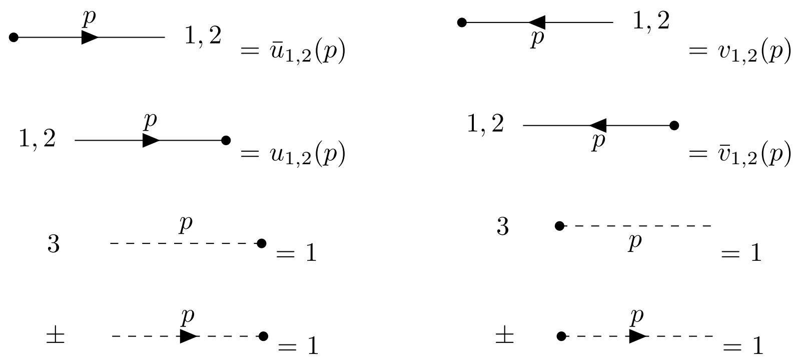


$= -i\sqrt{2}g$



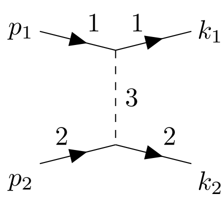
$= -i\sqrt{2}g$

The external lines are just the usual fermionic and scalar lines:

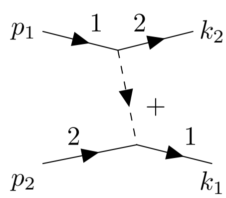


(4) For $f_1 f_2 \to f_1 f_2$, we have the following two processes:

(1)

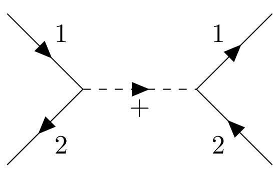


(2)

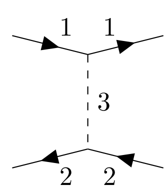


For $f_1 \bar{f}_2 \to f_1 \bar{f}_2$ we have the following two processes:

(3)



(4)



(5) We take the fermion masses to go to zero, but the scalar masses to remain. For the neutral current interaction (diagram (1)), we have:

$$\mathcal{M}_1 = ig^2 \bar{u}_2(k_2) u_2(p_2) \frac{1}{(p_1 - k_1)^2 - M^2} \bar{u}_1(k_1) u_1(p_1). \tag{12}$$

For the charged current diagram (diagram (2)), we have:

$$\mathcal{M}_2 = -2ig^2 \bar{u}_1(k_1) u_2(p_2) \frac{1}{(p_1 - k_2)^2 - M^2} \bar{u}_2(k_2) u_1(p_1). \tag{13}$$

The total, unpolarised, spin-averaged amplitude is given by

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}_1|^2+|\mathcal{M}_2|^2+\mathcal{M}_1\mathcal{M}_2^*+\mathcal{M}_1^*\mathcal{M}_2. \tag{14}$$

Let's calculate these four contributions separately:

$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}_1|^2 = \frac{g^4}{4}\sum_{s_1,s_2,s_3,s_4}\frac{1}{((p_1-k_1)^2-M^2)^2}$$

$$\times\, \bar{u}_2^{s_4}(k_2)u_2^{s_2}(p_2)\bar{u}_1^{s_3}(k_1)u_1^{s_1}(p_1)u_2^{s_4}(k_2)\bar{u}_2^{s_2}(p_2)u_1^{s_3}(k_1)\bar{u}_1^{s_1}(p_1) \tag{15}$$

$$= \frac{g^4}{4(t-M^2)^2}\mathrm{Tr}[\not{k}_2\not{p}_2]\mathrm{Tr}[\not{k}_1\not{p}_1] \tag{16}$$

$$= \frac{4g^4}{(t-M^2)^2}(k_2\cdot p_2)(k_1\cdot p_1). \tag{17}$$

A similar calculation yields for the other three terms:

$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}_2|^2 = \frac{16g^4}{(u-M^2)^2}(k_2\cdot p_1)(k_1\cdot p_2), \tag{18}$$

$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}\mathcal{M}_1\mathcal{M}_2^* = \frac{1}{4}\sum_{s_1,s_2,s_3,s_4}\mathcal{M}_2\mathcal{M}_1^* \tag{19}$$

$$= \frac{-2g^4}{(t-M^2)(u-M^2)}\left((k_2\cdot p_2)(k_1\cdot p_1)-(k_2\cdot k_1)(p_2\cdot p_1)+(k_2\cdot p_1)(p_2\cdot k_1)\right), \tag{20}$$

where $t=(p_1-k_1)^2$ and $u=(p_1-k_2)^2$, and we have made use of the trace identities of the gamma matrices. Putting everything together, we obtain

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = g^4\Big[\frac{4(k_2\cdot p_2)(k_1\cdot p_1)}{(t-M^2)^2}+\frac{16(k_2\cdot p_1)(k_1\cdot p_2)}{(u-M^2)^2} \tag{21}$$

$$-\frac{4}{(t-M^2)(u-M^2)}((k_2\cdot p_2)(k_1\cdot p_1)-(k_2\cdot k_1)(p_2\cdot p_1)+(k_2\cdot p_1)(p_2\cdot k_1))\Big] \tag{22}$$

(6) For writing everything in terms of Mandelstams, the following relations are useful:

$$s=(p_1+p_2)^2=2p_1\cdot p_2=2k_1\cdot k_2\to p_1\cdot p_2=k_1\cdot k_2=s/2 \tag{23}$$

$$t=(p_1-k_1)^2=-2p_1\cdot k_1=-2p_2\cdot k_2\to p_1\cdot k_1=p_2\cdot k_2=-t/2 \tag{24}$$

$$u=(p_1-k_2)^2=-p_1\cdot k_2=-2p_2\cdot k_1\to p_1\cdot k_2=p_2\cdot k_1=-u/2 \tag{25}$$

The amplitude can then easily be rewritten as

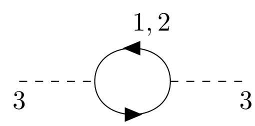
$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{4g^4}{(t-M^2)^2}\frac{t^2}{4} + \frac{16g^4}{(u-M^2)^2}\frac{u^2}{4} - \frac{4g^2}{(t-M^2)(u-M^2)}\left(\frac{t^2}{4} - \frac{s^2}{4} + \frac{u^2}{4}\right). \tag{26}$$

(7) In the limit of large scalar mass, the amplitude reduces to

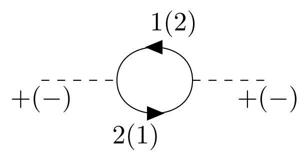
$$\lim_{M\to\infty}|\mathcal{M}|^2 = \frac{4g^2}{M^4}\frac{t^2}{4} + \frac{16g^2}{M^4}\frac{u^2}{4} - \frac{4g^4}{M^4}\left(\frac{t^2}{4} - \frac{s^2}{4} + \frac{u^2}{4}\right) \tag{27}$$

$$= \frac{g^4}{M^4}(3u^2 + s^2) \tag{28}$$

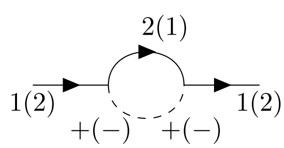
(8) We are asked to draw all 1-loop corrections to the two-point functions (i.e. the propagators) and compute their divergence. This is $D = 4 - 2P_s - P_f$.



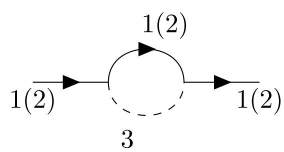
$D = 4 - 1 \cdot 2 = 2$



$D = 4 - 1 \cdot 2 = 2$



$D = 4 - 2 \cdot 1 - 1 \cdot 1 = 1$



$D = 4 - 2 \cdot 1 - 1 \cdot 1 = 1$

(9) The Lagrangian has an U(1) symmetry, corresponding to the conservation of fermion number, and a SU(2) symmetry corresponding to a rotation of all the fields.

The U(1) symmetry is under the transformation

$$\Psi' = e^{i\phi}\Psi \tag{29}$$

and the associate Noether current is

$$j^\mu = \bar{\psi}_1\gamma^\mu\psi_1 + \bar{\psi}_2\gamma^\mu\psi_2. \tag{30}$$

In order to expose the SU(2) symmetry, consider first a SU(2) transformation of the fermion doublet:

$$\Psi' = \Lambda_{\vec{\theta}}\Psi = e^{\frac{i}{2}\vec{\theta}\cdot\sigma}\Psi. \tag{31}$$

where $\vec{\theta} = \theta\hat{n}$ is a three vector of length θ and $\hat{n}$ is a unimodular vector.

Upon this transformation, the interaction term, using its form Eq. (3), transforms according to

$$\bar{\Psi}'\vec{\sigma}\Psi' \cdot \vec{\phi} = \bar{\Psi}\Lambda_{\vec{\theta}}^{-1}\vec{\sigma}\Lambda_{\vec{\theta}}\Psi \cdot \vec{\phi} \tag{32}$$

where we have used a three-vector notation for the Pauli matrices and the scalar fields. Using the properties of the Pauli matrices, it is easy to see that

$$\Lambda_{\vec{\theta}}^{-1}\sigma^i\Lambda_{\vec{\theta}} = R^i_{\vec{\theta}j}\sigma^j, \tag{33}$$

where ${R_{\vec{\theta}}}^i{}_j$ is a rotation of angle θ about the axis $\hat{n}$. It immediately follows that the interaction term is invariant if the scalar fields also transform as

$$\phi'^i = {R^{-1}}^i{}_j\phi^j. \tag{34}$$

In order to determine the Noether current we write the infintesimal form of the transformations Eq. (33,32)

$$\Psi' = \left(1 + \frac{i}{2}\vec{\theta}\cdot\sigma\right)\Psi + O(\theta^2) \tag{35}$$

$$\phi'_i = \left(1 + \vec{\theta}\cdot\vec{M}_{ij}\right)\phi^j + O(\theta^2), \tag{36}$$

where ${M^a}^i{}_j$ is the generator of rotations about the a-th axis. We then immedtately find that the Noether current is given by

$$J^\mu_a = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\delta^a\phi^i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^i)}\delta^a\psi^i \tag{37}$$

$$= \bar{\Psi}\sigma^a\gamma^\mu\Psi + i\partial^\mu\phi_i(M^a)^i_j\phi_j. \tag{38}$$