(A) EXERCISES

Exercise 2.1. By using only the canonical commutation relations (2.1), show that the space-time "charges", P^α, associated with the translation current $\theta_c{}^{\mu\alpha}$, $P^\alpha = \int d^3x\, \theta_c{}^{0\alpha}(t, \mathbf{x})$, generate the correct transformation on the fields.

$$i[P^\alpha, \phi(x)] = \partial^\alpha \phi(x) = \delta_T^\alpha \phi(x) .$$

Exercise 2.2. Consider Lorentz transformations.

$$\delta_L^{\alpha\beta} \phi(x) = (x^\alpha \partial^\beta - x^\beta \partial^\alpha + \Sigma^{\alpha\beta})\, \phi(x) .$$

Here $\Sigma^{\alpha\beta}$ is the spin matrix appropriate to the field $\phi(x)$. It specifies the representation of Lorentz group under which $\phi(x)$ transforms. $\Sigma^{\alpha\beta} = 0$ for spin zero bosons; $\Sigma_{(ij)}{}^{\alpha\beta} = \frac{1}{2} i \sigma_{ij}{}^{\alpha\beta}$ for spin $\frac{1}{2}$ fermions; ij are the Dirac indices appropriate to a 4-component fermion field and $\sigma^{\alpha\beta} = [\gamma^\alpha, \gamma^\beta]/2i$; $\Sigma_{(\mu\nu)}{}^{\alpha\beta} = g_\mu^\alpha g_\nu^\beta - g_\nu^\alpha g_\mu^\beta$ for vector bosons ($\mu\nu$ are the space-time indices appropriate to a vector field). Under what conditions, on a translationally invariant $\mathscr{L}$, is the above a symmetry transformation of the theory? Show that the conserved, canonical space-time current appropriate to Lorentz transformation is

$$M_c{}^{\mu\alpha\beta}(x) = x^\alpha \theta_c{}^{\mu\beta}(x) - x^\beta \theta_c{}^{\mu\alpha}(x) + \pi^\mu(x) \Sigma^{\alpha\beta} \phi(x) .$$

By use of the canonical commutators (2.1), verify that the "charges"

$$M^{\alpha\beta} = \int d^3x\, M_c{}^{0\alpha\beta}(t, \mathbf{x}) ,$$

generate the correct transformations on the fields.

$$i[M^{\alpha\beta}, \phi(x)] = \delta_L^{\alpha\beta} \phi(x) .$$

Exercise 2.3. Consider the Belinfante tensor $\theta_B{}^{\mu\alpha}$ defined in (2.17). Show that the Belinfante tensor is conserved when $\theta_c{}^{\mu\alpha}$ is. Next show that the charge P^α, defined in Exercise 2.1, are the same regardless whether they are constructed from $\theta_B{}^{\mu\alpha}$ or $\theta_c{}^{\mu\alpha}$. Therefore for purposes of describing translations, $\theta_B{}^{\mu\alpha}$ may be used instead of $\theta_c{}^{\mu\alpha}$. $\theta_B{}^{\mu\alpha}$ has additional advantages. With the help of the equations of motion, as well as the condition for Lorentz covariance, derived in Exercise 2.2, show that $\theta_B^{\mu\alpha}$ is symmetric in μ and α, while $\theta_c^{\mu\alpha}$ has this property only for spinless fields $\Sigma^{\alpha\beta} = 0$.

The Belinfante tensor takes on a significance over the canonical tensor in connection with the space-time current associated with Lorentz transformation. Consider

$$M_B^{\mu\alpha\beta}(x) = x^\alpha \theta_B^{\mu\beta}(x) - x^\beta \theta_B^{\mu\alpha}(x) .$$

Show that $M_B^{\mu\alpha\beta}(x)$ is conserved when $M_c^{\mu\alpha\beta}(x)$, defined in Exercise 2.2, possesses this property. Also show that $M_B^{\mu\alpha\beta}(x)$, leads to the same charges $M^{\alpha\beta}$ as $M_c^{\mu\alpha\beta}(x)$ does. Hence $M_B^{\mu\alpha\beta}$ can be used instead of $M_c^{\mu\alpha\beta}$ as the space-time current for Lorentz transformations; however $M_B^{\mu\alpha\beta}$ is much simpler than $M_c^{\mu\alpha\beta}$, since the former contains no explicit spin term $\pi^\mu \Sigma^{\alpha\beta} \phi$. The full significance of the Belinfante tensor emerges when one asserts that it is this energy-momentum tensor (rather than any other one) to which gravitons couple in the Einstein theory of gravity, general relativity.

Exercise 2.4. By use of canonical expressions for θ_c^{00} and J_0^a, as well as the canonical commutators, derive the equal-time commutator between these two operators, given in (2.18). Verify that the answer remains unchanged when θ_c^{00} is replaced by θ_B^{00}.

Exercise 2.5. Consider the scalar field Lagrangian.

$$\mathscr{L}(x) = \tfrac{1}{2}\phi^\mu(x)\phi_\mu(x) + gx^2\phi(x) .$$

Construct the energy momentum tensor from the formula (2.11), and show that it is not conserved. Evaluate the commutator $[P^0(t), P^i(t)]$ and verify that it does not vanish. (When translation invariance holds this commutator vanishes.) Construct the Lorentz current for this model, and verify that it is conserved.

Exercise 2.6. Verify that (2.24) is the most general solution of (2.23b). Hint: Multiply (2.23b) by y_i and integrate over $\mathbf{y}$.

Exercise 2.7. Consider the ETC of electromagnetic currents which hold in scalar electrodynamics.

$$[J^0(0,\mathbf{x}), J^0(0)] = 0 ,$$

$$[J^0(0,\mathbf{x}), J^i(0)] = S(0)\,\partial^i\delta(\mathbf{x}) ,$$

$$[J^i(0,\mathbf{x}), J^j(0)] = 0 .$$

Here $S(y)$ is a scalar operator. Rewrite this commutator in a Lorentz covariant, but frame-dependent fashion by introducing the unit time-like vector n^μ. Consider also

$$\mathrm{T} J^\mu(x) J^\nu(0) \equiv \mathrm{T}^{\mu\nu}(x;n) .$$

Show that this is not covariant, and construct a covariantizing seagull $\tau^{\mu\nu}(x;n)$. Determine the seagull by requiring that $\mathrm{T}^{*\mu\nu}(x) = \mathrm{T}^{\mu\nu}(x;n) + \tau^{\mu\nu}(x;n)$ be conserved.

Exercise 3.1. Consider the free boson propagator $D(q) = i/(q^2 - m^2)$. By use of the BJL theorem verify the canonical commutation relations.

$$i[\phi(0,\mathbf{x}), \phi(0)] = i[\dot{\phi}(0,\mathbf{x}), \dot{\phi}(0)] = 0 ,$$

$$i[\dot{\phi}(0,\mathbf{x}), \phi(0)] = \delta(\mathbf{x}) .$$

Next consider the full propagator of renormalized fields. It may be written in the form

$$G(q) = i \int_0^\infty \mathrm{d}a^2 \, \frac{\rho(a^2)}{q^2 - a^2} .$$

The spectral function ρ can be shown to be non-negative. (A renormalized field $\tilde{\phi}$ is proportionally related to the unrenormalized field ϕ by $\phi = Z^{1/2} \tilde{\phi}$. In perturbation theory Z is cutoff dependent.) What can you deduce about the vacuum expectation of the canonical commutators in the complete theory?

Exercise 3.2. Consider the vacuum polarization tensor, which (formally) can be written as

$$\mathrm{T}^{*\mu\nu}(q) = \int \mathrm{d}^4x \, e^{iq\cdot x} \langle 0 | \mathrm{T}^* J^\mu(x) J^\nu(0) | 0 \rangle$$

$$= (g^{\mu\nu} q^2 - q^\mu q^\nu) \int_0^\infty \mathrm{d}a^2 \, \frac{\sigma(a^2)}{q^2 - a^2} .$$

It can be shown that $\sigma(a^2)$ has a definite sign. Extract the T product from $\mathrm{T}^{*\mu\nu}(q)$ and calculate $\langle 0 | [J^0(0,\mathbf{x}), J^i(0)] | 0 \rangle$ in terms of $\sigma(a^2)$. (Do not concern yourselves here with problems of convergence; these will be discussed in Exercise 6.1.)

Exercise 4.1. Show that

$$\Delta^{\mu}(a) = i\int \frac{d^4 r}{(2\pi)^4}\left[\frac{r^{\mu}+a^{\mu}}{([r+a]^2-m^2)^2} - \frac{r^{\mu}}{(r^2-m^2)^2}\right]$$

$$= \frac{-a^{\mu}}{32\pi^2} \quad .$$

This verifies (4.19b).

Exericse 4.2. Show that

$$\Delta^{\alpha\mu\nu}(p,q\,|a)$$

$$= i\int \frac{d^4 r}{(2\pi)^4}\,\mathrm{Tr}\,\gamma^5\gamma^{\alpha}\Big\{[r_\beta\gamma^\beta + a_\beta\gamma^\beta + p_\beta\gamma^\beta - m]^{-1}$$

$$\times\ \gamma^{\mu}[r_\beta\gamma^\beta + a_\beta\gamma^\beta - m]^{-1}\gamma^{\nu}[r_\beta\gamma^\beta + a_\beta\gamma^\beta + q_\beta\gamma^\beta - m]^{-1}$$

$$- [r_\beta\gamma^\beta + p_\beta\gamma^\beta - m]^{-1}\gamma^{\mu}[r_\beta\gamma^\beta - m]^{-1}\gamma^{\nu}[r_\beta\gamma^\beta - q_\beta\gamma^\beta - m]^{-1}\Big\}$$

$$= \frac{-1}{8\pi^2}\,\epsilon^{\alpha\mu\nu\beta}\,a_\beta \quad .$$

This verifies (4.21).

Exercise 4.3. Show that

$$i\int \frac{d^4 r}{(2\pi)^4}\,\mathrm{Tr}\,\gamma^5\gamma^{\alpha}\Big\{[r_\beta\gamma^\beta + q_\beta\gamma^\beta - m]^{-1}\gamma^{\nu}[r_\beta\gamma^\beta - p_\beta\gamma^\beta - m]^{-1}$$

$$- [r_\beta\gamma^\beta + p_\beta\gamma^\beta - m]^{-1}\gamma^{\nu}[r_\beta\gamma^\beta - q_\beta\gamma^\beta - m]^{-1}\Big\}$$

$$= \frac{1}{4\pi^2}\,\epsilon^{\alpha\mu\nu\beta}\,p_\mu q_\beta \quad .$$

This verifies (4.25c).

Exercise 5.1. Compute canonically the symmetric part of

$$\int d^3x\, [\dot{J}^i(0,\mathbf{x}), J^j(0)] \ ,$$

in the quark-vector gluon model, where

$$i\gamma_\mu \partial^\mu \psi = m\psi - g\gamma^\mu \psi B_\mu \ .$$

Verify that the spin averaged, matrix element of that commutator, between diagonal proton states is of the form

$$A(\delta^{ij}\mathbf{p}^2 - p^i p^j) + B\delta^{ij} \ .$$

Exercise 5.2. Assume that the dispersion relation for T_L needs one subtraction, performed at $\nu = 0$, $(\omega = \infty)$. Show that the ST sum rule reduces in that instance to an uninformative relation between the subtraction term and the ST.

Exercise 6.1. When the spectral function $\sigma(a^2)$, relevant to the vacuum polarization tensor $T^{*\mu\nu}$, does not vanish as $a^2 \to \infty$, the dispersive representation for $T^{*\mu\nu}$ given in Exercise 3.2 will not converge. Assuming that

$$\lim_{a^2 \to \infty} \sigma(a^2) = A \ ,$$

$$\lim_{a^2 \to \infty} a^2 \Big(\sigma(a^2) - A\Big) = B \ ,$$

the following subtracted form for $T^{*\mu\nu}$ may be established

$$T^{*\mu\nu}(q) = \int d^4x\, e^{iq\cdot x} \langle 0|T^* J^\mu(x) J^\nu(0)|0\rangle$$

$$= (q^{\mu\nu}q^2 - q^\mu q^\nu)\left[C + q^2 \int_{4m^2}^{\infty} da^2 \frac{\sigma(a^2)}{a^2(q^2 - a^2)}\right] \ .$$

Here C is a subtraction constant, and $4m^2$ is the appropriate threshold for the dispersive integral. Calculate the vacuum expectation value of the $[J^0, J^i]$ ETC, and show that a triple derivative of the δ function is present. Hint: Let $\tilde{\sigma}(a^2) = \sigma(a^2) - A - B/a^2$.

Exercise 6.2. Consider $\mathrm{T}(p,q)$, defined by

$$\mathrm{T}(p,q) = \int \mathrm{d}^4x\,\mathrm{d}^4y\, e^{ip\cdot x}\, e^{iq\cdot y} \langle \alpha | \mathrm{T}A(x)B(y)C(0)|\beta\rangle \;,$$

where α and β are arbitrary states. Show that

$$\lim_{q_0\to\infty}\;\lim_{p_0\to\infty}\; -p_0q_0\mathrm{T}(p,q)$$

$$= \int \mathrm{d}^3x\,\mathrm{d}^3y\, e^{-i\mathbf{p}\cdot\mathbf{x}}\, e^{-i\mathbf{q}\cdot\mathbf{y}} \langle \alpha | [B(0,\mathbf{y}), [A(0,\mathbf{x}), C(0)]]\,|\beta\rangle \;.$$

What is the result if the limit is performed on opposite order,

$$\lim_{p_0\to\infty}\;\lim_{q_0\to\infty}\; -p_0q_0\mathrm{T}(p,q)\;?$$

Exercise 7.1. Show that the invariant functions $\widetilde{F}_i$, defined from

$$C^{\mu\nu}(q,p) = \int \frac{\mathrm{d}^4x}{(2\pi)^4} e^{iq\cdot x} \langle p | [J^\mu(x), J^\nu(0)]\,|p\rangle$$

$$= -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\widetilde{F}_1$$

$$+\left(p^\mu - q^\mu\frac{p\cdot q}{q^2}\right)\left(p^\nu - q^\nu\frac{p\cdot q}{q^2}\right)\frac{\widetilde{F}_2}{p\cdot q} \;,$$

are dimensionless. The state $|p\rangle$ is normalized covariantly.

Exercise 7.2. Compute $i[D(t), P^\mu]$ and $i[D(t), M^{\mu\nu}]$, where $D(t)$ is the dilatation "charge".

Exercise 7.3. By use of canonical ETC, show that

$$i[D(t), \phi(x)] = \delta_{\mathrm{D}}\,\phi(x) \;,$$

$$i[K^\alpha(t), \phi(x)] = \delta_{\mathrm{C}}{}^\alpha\,\phi(x) \;.$$

Here $D(t)$ and $K^\alpha(t)$ are dilatation and conformal "charges" respectively. Hint: Use the canonical formulas for the currents.

Exercise 7.4. Consider a scalar field Lagrangian $\mathscr{L}$ depending on ϕ and $\partial^\mu\phi$. What is the most general form for $\mathscr{L}$ which is scale invariant? What is the most general form that is conformly invariant? Derive $\theta_c^{\mu\nu}$ for this conformly invariant theory. Since this is a spinless theory, $\theta_c^{\mu\nu} = \theta_B^{\mu\nu}$. What is $\theta^{\mu\nu}$?

(B) SOLUTIONS

Exercise 2.1. It is clear that the argument of $\phi(x)$ is inessential. Hence we consider

$$i[P^\alpha, \phi(0)] = \int d^3y\; i[\theta_c^{0\alpha}(0, y), \phi(0)] \; .$$

(a) $\alpha = i$

$$\begin{aligned} i[P^i, \phi(0)] &= \int d^3y\; i[\pi^0(0, \mathbf{y})\,\phi^i(0, \mathbf{y}), \phi(0)] \\ &= \int d^3y\; i[\pi^0(0, \mathbf{y})\,\phi(0)]\; \phi^i(0, \mathbf{y}) \\ &= \int d^3y\; \delta(\mathbf{y})\,\phi^i(0, \mathbf{y}) \\ &= \phi^i(0) \; , \end{aligned}$$

(b) $\alpha = 0$

$$\begin{aligned} i[P^0, \phi(0)] &= \int d^3y\; i[\pi^0(0, \mathbf{y})\phi^0(0, \mathbf{y}) - \mathscr{L}(0, \mathbf{y}), \phi(0)] \\ &= \phi^0(0) + \int d^3y\; \Big(\pi^0(0, \mathbf{y})\, i[\phi^0(0, \mathbf{y}), \phi(0)] \\ &\qquad\qquad - i[\mathscr{L}(0, \mathbf{y}), \phi(0)]\Big) \; . \end{aligned}$$

We now show that the second term in the above is zero. To do this, we introduce the technique of functional differential with respect to $\phi(0)$ and $\pi^0(0)$. By definition,

$$\frac{\delta'\phi(0, \mathbf{x})}{\delta'\phi(0)} = \delta(\mathbf{x}) \; , \qquad \frac{\delta'\phi^i(0, \mathbf{x})}{\delta'\phi(0)} = \partial^i\delta(\mathbf{x}) \; ,$$

$$\frac{\delta'\pi^0(0,\mathbf{x})}{\delta'\phi(0)} = 0 \;, \qquad \frac{\delta'\phi(0,\mathbf{x})}{\delta'\pi^0(0)} = 0 \;,$$

$$\frac{\delta'\partial^i\phi(0,\mathbf{x})}{\delta'\pi^0(0)} = 0 \;, \qquad \frac{\delta'\pi^0(0,\mathbf{x})}{\delta'\pi^0(0)} = \delta(\mathbf{x}) \;.$$

The quantity ϕ^0 is considered to be a functional of both π^0 and ϕ. For any functions F which depends on both ϕ and ϕ^μ, the chain rule holds

$$\frac{\delta' F(0,\mathbf{x})}{\delta'\phi(0)} = \frac{\delta F(0,\mathbf{x})}{\delta\phi}\delta(\mathbf{x}) + \frac{\delta F(0,\mathbf{x})}{\delta\phi^i}\partial^i\delta(\mathbf{x})$$

$$+ \frac{\delta F(0,\mathbf{x})}{\delta\phi^0}\,\frac{\delta'\phi^0(0,\mathbf{x})}{\delta'\phi(0)}$$

$$\frac{\delta' F(0,\mathbf{y})}{\delta'\pi^0(0)} = \frac{\delta F(0,\mathbf{x})}{\delta\phi^0}\,\frac{\delta'\phi^0(0,\mathbf{x})}{\delta'\pi^0(0)} \;.$$

Here δ indicates the ordinary variational derivative; δ' is the functional derivative. Applying this formalism, gives

$$\pi^0(0,\mathbf{y})\,i[\phi^0(0,\mathbf{y}),\phi(0)] - i[\mathscr{L}(0,\mathbf{y}),\phi(0)]$$

$$= \pi^0(0,\mathbf{y})\frac{\delta'\phi^0(0,\mathbf{y})}{\delta'\pi^0(0)} - \frac{\delta'\mathscr{L}(0,\mathbf{y})}{\delta'\pi^0(0)}$$

$$= \pi^0(0,\mathbf{y})\frac{\delta'\phi^0(0,\mathbf{y})}{\delta'\pi^0(0)} - \frac{\delta\mathscr{L}(0,\mathbf{y})}{\delta\phi^0}\,\frac{\delta'\phi^0(0,\mathbf{y})}{\delta'\pi^0(0)} = 0 \;.$$

The definition of π^0 was used: $\pi^0 = \delta\mathscr{L}/\delta\phi^0$.

(c) Note that we have established the stronger result

$$i[\theta_c^{0\alpha}(t,\mathbf{y}),\phi(t,\mathbf{x})] = \phi^\alpha(x)\,\delta(\mathbf{x}-\mathbf{y}) \;.$$

Exercise 2.2.

(a)
$$\delta_{\mathrm{L}}^{\alpha\beta}\mathscr{L} = \pi_\mu \delta_{\mathrm{L}}^{\alpha\beta}\phi^\mu + \frac{\delta\mathscr{L}}{\delta\phi}\delta_{\mathrm{L}}^{\alpha\beta}\phi$$

$$= \pi_\mu \partial^\mu \delta_{\mathrm{L}}^{\alpha\beta}\phi + \frac{\delta\mathscr{L}}{\delta\phi}\delta_{\mathrm{L}}^{\alpha\beta}\phi$$

$$= \pi_\mu \left[g^{\mu\alpha}\phi^\beta - g^{\mu\beta}\phi^\alpha + (x^\alpha\partial^\beta - x^\beta\partial^\alpha)\phi^\mu + \Sigma^{\alpha\beta}\phi^\mu\right]$$

$$+ \frac{\delta\mathscr{L}}{\delta\phi}(x^\alpha\partial^\beta\phi - x^\beta\partial^\alpha\phi) + \frac{\delta\mathscr{L}}{\delta\phi}\Sigma^{\alpha\beta}\phi \;.$$

By translation invariance, the above may be written as

$$\delta^{\alpha\beta}\mathscr{L} = x^\alpha\partial^\beta\mathscr{L} - x^\beta\partial^\alpha\mathscr{L} + \pi_\mu\Sigma^{\alpha\beta}\phi^\mu + \frac{\delta\mathscr{L}}{\delta\phi}\Sigma^{\alpha\beta}\phi + \pi^\alpha\phi^\beta$$

$$- \pi^\beta\phi^\alpha \;,$$

$$= \partial_\mu[g^{\mu\beta}x^\alpha\mathscr{L} - g^{\mu\alpha}x^\beta\mathscr{L}] + \pi_\mu\Sigma^{\alpha\beta}\phi^\mu + \frac{\delta\mathscr{L}}{\delta\phi}\Sigma^{\alpha\beta}$$

$$+ \pi^\alpha\phi^\beta - \pi^\beta\phi^\alpha \;.$$

Hence the condition for Lorentz covariance of the theory is

$$\pi_\mu\Sigma^{\alpha\beta}\phi^\mu + \frac{\delta\mathscr{L}}{\delta\phi}\Sigma^{\alpha\beta}\phi = \pi^\beta\phi^\alpha - \pi^\alpha\phi^\beta \;.$$

(b) In the notation of Section 2,

$$\Lambda^{\mu\alpha\beta} = x^\alpha g^{\mu\beta}\mathscr{L} - x^\beta g^{\mu\alpha}\mathscr{L} \;,$$

and the canonical conserved current is

$$M_{\mathrm{c}}^{\mu\alpha\beta} = \pi^\mu\delta_{\mathrm{L}}^{\alpha\beta}\phi - \Lambda^{\mu\alpha\beta}$$

$$= \pi^\mu(x^\alpha\phi^\beta - x^\beta\phi^\alpha + \Sigma^{\alpha\beta}\phi) - x^\alpha g^{\mu\beta}\mathscr{L} - x^\beta g^{\mu\alpha}\mathscr{L}$$

$$= x^\alpha\theta_{\mathrm{c}}{}^{\mu\beta} - x^\beta\theta_{\mathrm{c}}{}^{\mu\alpha} + \pi^\mu\Sigma^{\alpha\beta}\phi \;.$$

(c) Consider

$$i[M_c^{0\alpha\beta}(t,\mathbf{y}),\ \phi(t,\mathbf{x})]$$

$$= iy^\alpha[\theta_c{}^{0\beta}(t,\mathbf{y}),\ \phi(t,\mathbf{x})] - iy^\beta[\theta_c{}^{0\alpha}(t,\mathbf{y}),\ \phi(t,\mathbf{x})]$$

$$+ i[\pi^0(t,\mathbf{y})\Sigma^{\alpha\beta}\phi(t,\mathbf{y}),\ \phi(t,\mathbf{x})] \ .$$

The commutators with $\theta_c{}^{0\alpha}$ are evaluated in Exercise 2.1. Hence

$$i[M_c^{0\alpha\beta}(t,\mathbf{y}),\ \phi(t,\mathbf{x})]$$

$$= (x^\alpha\partial^\beta\phi(x) - x^\beta\partial^\alpha\phi(x) + \Sigma^{\alpha\beta}\phi(x)\,\delta(\mathbf{x}-\mathbf{y}))$$

$$= \delta_L^{\alpha\beta}\phi(x)\,\delta(\mathbf{x}-\mathbf{y}) \ .$$

The desired result now follows.

Exercise 2.3.

(a) $\theta_B{}^{\mu\alpha} = \theta_c{}^{\mu\alpha} + \frac{1}{2}\partial_\lambda X^{\lambda\mu\alpha}$,

$$X^{\lambda\mu\alpha} = -X^{\mu\lambda\alpha} = \pi^\lambda\Sigma^{\mu\alpha}\phi - \pi^\mu\Sigma^{\lambda\alpha}\phi - \pi^\alpha\Sigma^{\lambda\mu}\phi \, .$$

Since $X^{\lambda\mu\alpha}$ is explicitly anti-symmetric in $\lambda\mu$, $\partial_\lambda X^{\lambda\mu\alpha}$ does not contribute to the divergence of $\theta_B{}^{\mu\alpha}$ in μ, nor does it contribute to the charges $\int d^3x\,\theta_B{}^{0\alpha}(x)$.

(b) $\theta_B{}^{\mu\alpha} = \pi^\mu\phi^\alpha - g^{\mu\alpha}\mathscr{L}$

$$+ \tfrac{1}{2}\partial_\lambda[\pi^\lambda\Sigma^{\mu\alpha}\phi - \pi^\mu\Sigma^{\lambda\alpha}\phi - \pi^\alpha\Sigma^{\lambda\mu}\phi] \ .$$

The only terms that are not explicitly symmetric in $\mu\alpha$ are

$$\pi^\mu\phi^\alpha + \frac{1}{2}\partial_\lambda[\pi^\lambda\Sigma^{\mu\alpha}\phi]$$

$$= \pi^\mu\phi^\alpha + \frac{1}{2}\partial_\lambda\pi^\lambda\Sigma^{\mu\alpha}\phi + \frac{1}{2}\pi^\lambda\Sigma^{\mu\alpha}\phi_\lambda \ ,$$

$$= \pi^\mu\phi^\alpha + \frac{1}{2}\frac{\delta\mathscr{L}}{\delta\phi}\Sigma^{\mu\alpha}\phi + \frac{1}{2}\pi^\lambda\Sigma^{\mu\alpha}\phi_\lambda \ .$$

The equations of motion have been used. Next use the constraint on $\mathscr{L}$ imposed by Lorentz covariance; see Exercise 2.2. We have

$$\begin{aligned}\pi^\mu\phi^\alpha + \tfrac{1}{2}\partial_\lambda[\pi^\lambda\Sigma^{\mu\alpha}\phi] &= \pi^\mu\phi^\alpha - \tfrac{1}{2}\pi^\mu\phi^\alpha + \tfrac{1}{2}\pi^\alpha\phi^\mu \\ &= \tfrac{1}{2}\pi^\mu\phi^\alpha + \tfrac{1}{2}\pi^\alpha\phi^\mu .\end{aligned}$$

(c)
$$\begin{aligned}M_B{}^{\mu\alpha\beta} &= x^\alpha\theta_B{}^{\mu\beta} - x^\beta\theta_B{}^{\mu\alpha} \\ &= x^\alpha\theta_c{}^{\mu\beta} - x^\beta\theta_c{}^{\mu\alpha} + x^\alpha\tfrac{1}{2}\partial_\lambda X^{\lambda\mu\beta} - x^\beta\tfrac{1}{2}\partial_\lambda X^{\lambda\mu\alpha} \\ &= M_c{}^{\mu\alpha\beta} + \tfrac{1}{2}\partial_\lambda[X^{\lambda\mu\beta}x^\alpha - X^{\lambda\mu\alpha}x^\beta] \\ &\quad + \tfrac{1}{2}X^{\beta\mu\alpha} - \tfrac{1}{2}X^{\alpha\mu\beta} - \pi^\mu\Sigma^{\alpha\beta}\phi .\end{aligned}$$

From the explicit form for $X^{\lambda\mu\alpha}$, we see that the last three terms cancel. The total derivative term is the divergence in λ of a tensor which is antisymmetric in $\mu\lambda$; hence that object does not contribute to the divergence in μ, nor to charges.

Exercise 2.4.

(a)
$$\begin{aligned}i[\theta_c{}^{00}(t,\mathbf{x}), J_0{}^a(t,\mathbf{y})] &= i[\theta_c{}^{00}(t,\mathbf{x}), \pi^0(t,\mathbf{y})T^a\phi(t,\mathbf{y})] \\ &= \pi^0(t,\mathbf{y})T^a\, i[\theta_c{}^{00}(t,\mathbf{x}), \phi(t,\mathbf{y})] \\ &\quad + i[\theta_c{}^{00}(t,\mathbf{x}), \pi^0(t,\mathbf{y})]\, T^a\phi(t,\mathbf{y}) .\end{aligned}$$

The first commutator is evaluated as in Exercise 2.1. The second is expressed in terms of functional derivatives. The result is

$$\begin{aligned}i[\theta_c{}^{00}(t,\mathbf{x}), J_0{}^a(t,\mathbf{y})] &= \pi^0(t,\mathbf{y})T^a\phi^0(t,\mathbf{y})\,\delta(\mathbf{x}-\mathbf{y}) \\ &\quad - \frac{\delta'\theta_c{}^{00}(t,\mathbf{x})}{\delta'\phi(t,\mathbf{y})}\, T^a\phi(t,\mathbf{y}) .\end{aligned}$$

The functional derivative is now calculated from the formula for $\theta_c{}^{00}(t,\mathbf{x})$.

$$\frac{\delta'\theta_c{}^{00}(t,\mathbf{x})}{\delta'\phi(t,\mathbf{y})} = \pi^0(t,\mathbf{x})\frac{\delta'\phi^0(t,\mathbf{x})}{\delta'\phi(t,\mathbf{y})} - \frac{\delta'\mathscr{L}(t,\mathbf{x})}{\delta'\phi(t,\mathbf{y})} ,$$

$$\frac{\delta' \mathscr{L}(t,\mathbf{x})}{\delta' \phi(t,\mathbf{y})} = \frac{\delta \mathscr{L}(t,\mathbf{x})}{\delta \phi} \delta(\mathbf{x}-\mathbf{y}) + \frac{\delta \mathscr{L}(t,\mathbf{x})}{\delta \phi^0} \frac{\delta' \phi^0(t,\mathbf{x})}{\delta' \phi(t,\mathbf{y})}$$

$$+ \frac{\delta \mathscr{L}(t,\mathbf{x})}{\delta \phi^i} \partial^i \delta(\mathbf{x}-\mathbf{y})$$

$$= \partial_\lambda \pi^\lambda(t,\mathbf{x}) \delta(\mathbf{x}-\mathbf{y}) + \pi^0(t,\mathbf{x}) \frac{\delta' \phi^0(t,\mathbf{x})}{\delta' \phi(t,\mathbf{y})}$$

$$+ \pi_i(t,\mathbf{x}) \partial^i \delta(\mathbf{x}-\mathbf{y}) \ .$$

We have used the equations of motion and the definition of π^μ. It now follows that

$$\frac{\delta' \theta_c^{00}(t,\mathbf{x})}{\delta' \phi(t,\mathbf{y})} = -\partial_\lambda \pi^\lambda(t,\mathbf{x}) \delta(\mathbf{x}-\mathbf{y}) - \pi_i(t,\mathbf{x}) \partial^i \delta(\mathbf{x}-\mathbf{y}) \ ,$$

$$i[\theta_c^{00}(t,\mathbf{x}), J_0^a(t,\mathbf{y})]$$

$$= \pi^0(t,\mathbf{x}) T^a \phi^0(t,\mathbf{x}) \delta(\mathbf{x}-\mathbf{y}) + \partial_\lambda \pi^\lambda(t,\mathbf{x}) T^a \phi(t,\mathbf{x}) \delta(\mathbf{x}-\mathbf{y})$$

$$+ \pi^i(t,\mathbf{x}) T^a \phi(t,\mathbf{y}) \partial_i \delta(\mathbf{x}-\mathbf{y})$$

$$= \partial_0 [\pi^0(t,\mathbf{x}) T^a \phi(t,\mathbf{x})] \delta(\mathbf{x}-\mathbf{y})$$

$$+ \partial_i [\pi^i(t,\mathbf{x}) T^a \phi(t,\mathbf{x})] \delta(\mathbf{x}-\mathbf{y})$$

$$+ \pi^i(t,\mathbf{x}) T^a \phi(t,\mathbf{x}) \partial_i \delta(\mathbf{x}-\mathbf{y})$$

$$= \partial^\mu J_\mu^a(x) \delta(\mathbf{x}-\mathbf{y}) + J_i^a(x) \partial^i \delta(\mathbf{x}-\mathbf{y}) \ .$$

In the last formula we have used the definition for the current.

$$J_\mu^a = \pi_\mu T^a \phi \ .$$

(b) The difference between θ_B^{00} and θ_c^{00} is

$$\tfrac{1}{2}\partial_\lambda X^{\lambda 00} = \tfrac{1}{2}\partial_i X^{i00} = \partial_i[\pi^0 \Sigma^{0i}\phi] \ .$$

Since Σ^{0i} and T^a commute, (they operate in different spaces) $J_0{}^a$ commutes with $\frac{1}{2}\partial_i X^{i00}$.

Exercise 2.5.

(a)
$$\begin{aligned}\theta_c^{\mu\nu} &= \pi^\mu \phi^\nu - g^{\mu\nu}\mathscr{L}\\ &= \phi^\mu\phi^\nu - g^{\mu\nu}\mathscr{L}\end{aligned}$$

$$\begin{aligned}\partial_\mu\theta_c^{\mu\nu} &= \partial_\mu\phi^\mu\phi^\nu + \phi^\mu\partial_\mu\phi^\nu - \partial^\nu\mathscr{L}\\ &= gx^2\phi^\nu + \phi^\mu\partial_\mu\phi^\nu - \phi^\mu\partial^\nu\phi_\mu - 2gx^\nu\phi - gx^2\phi^\nu\\ &= -\,2gx^\nu\phi \ .\end{aligned}$$

We have used the equation of motion: $\partial_\mu\phi^\mu = gx^2$.

(b)
$$\begin{aligned} i[P^0(t),P^i(t)] &= \int d^3x\, d^3y\, i[\theta_c^{00}(t,\mathbf{x}),\theta_c^{0i}(t,\mathbf{y})]\\ &= \int d^3x\, d^3y\, i[\tfrac{1}{2}\phi^0(t,\mathbf{x})\phi^0(t,\mathbf{x}) - \tfrac{1}{2}\phi^j(t,\mathbf{x})\\ &\quad\times \phi_j(t,\mathbf{x}) - gx^2\phi(t,\mathbf{x}),\ \phi^0(t,\mathbf{y})\phi^i(t,\mathbf{y})]\\ &= \int d^3x\, d^3y\, \Big\{\phi^0(t,\mathbf{x})\phi^0(t,\mathbf{y})\frac{\partial}{\partial y_i}\delta(\mathbf{x}-\mathbf{y})\\ &\quad + \phi^j(t,\mathbf{x})\phi^i(t,\mathbf{y})\frac{\partial}{\partial x^j}\delta(\mathbf{x}-\mathbf{y})\\ &\quad + gx^2\phi^i(t,\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\Big\}\\ &= \int d^3x\, \big\{-\phi^0(t,\mathbf{x})\partial^i\phi^0(t,\mathbf{x}) + \phi^j(t,\mathbf{x})\partial_j\phi^i(t,\mathbf{x})\\ &\quad + gx^2\phi^i(t,\mathbf{x})\big\}\end{aligned}$$

$$= -\tfrac{1}{2}\int d^3x\,\partial^i\left\{\phi^0(t,\mathbf{x})\,\phi^0(t,\mathbf{x}) - \phi^j(t,\mathbf{x})\,\phi_j(t,\mathbf{x})\right.$$

$$\left. -\ 2gx^2\,\phi(t,\mathbf{x})\right\} - g\int d^3x\,x^i\,\phi(t,\mathbf{x})\ .$$

The first term is a surface integral; it may be dropped. We are left with

$$i[P^0(t), P^i(t)] = -g\int d^3x\,x^i\,\phi(t,\mathbf{x})\ .$$

(c) Since $\Sigma^{\alpha\beta}$ is zero for spin zero fields,

$$M_c{}^{\mu\alpha\beta}(x) = x^\alpha\,\theta_c{}^{\mu\beta} - x^\beta\,\theta_c{}^{\mu\alpha}$$

$$\partial_\mu M_c{}^{\alpha\beta}(x) = \theta_c{}^{\alpha\beta} + x^\alpha\,\partial_\mu\,\theta_c{}^{\alpha\beta} - \theta_c{}^{\beta\alpha} - x^\beta\,\partial_\mu\,\theta_c{}^{\mu\alpha}$$

$$= x^\alpha\,\partial_\mu\,\theta_c{}^{\mu\beta} - x^\beta\,\partial_\mu\,\theta_c{}^{\mu\alpha}\ .$$

We have used the symmetry of $\theta_c{}^{\mu\nu}$. From (a) we have

$$\partial_\mu M_c{}^{\mu\alpha\beta}(x) = 2g\,(x^\beta x^\alpha - x^\alpha x^\beta) = 0\ .$$

Exercise 2.6.

$$[J_0{}^b(t,\mathbf{y}), J_k^a(t,\mathbf{z})]\ \partial^k\,\delta(\mathbf{x}-\mathbf{z}) + [J_0{}^a(t,\mathbf{x}), J_k^b(t,\mathbf{z})]\ \partial^k\,\delta(\mathbf{z}-\mathbf{y})$$

$$= -\ f_{abc}\,\delta(\mathbf{x}-\mathbf{y})\,J_k^c(t,\mathbf{z})\,\partial^k\,\delta(\mathbf{z}-\mathbf{x})\ .$$

Multiplying by y_i and integrating over $\mathbf{y}$ gives

$$[J_0{}^a(t,\mathbf{x}), J_i^b(t,\mathbf{z})]$$

$$= -\ f_{abc}\,x_i\,J_k^c(t,\mathbf{z})\,\partial^k\,\delta(\mathbf{z}-\mathbf{x})$$

$$-\int d^3y\,y_i\,[J_0{}^b(t,\mathbf{y}), J_k^a(t,\mathbf{z})]\ \partial^k\,\delta(\mathbf{x}-\mathbf{z})$$

$$= -\ f_{abc}\,J_i^c(t,\mathbf{z})\,\delta(\mathbf{z}-\mathbf{x}) - f_{abc}\,z_i\,J_k^c(t,\mathbf{z})\,\partial^k\,\delta(\mathbf{z}-\mathbf{x})$$

$$-\int d^3y\,y_i\,[J_0{}^b(t,\mathbf{y}), J_k^a(t,\mathbf{z})]\ \partial^k\,\delta(\mathbf{x}-\mathbf{z})\ .$$