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2. Canonical and Space-Time Constraints in Current Algebra

2.1 *Canonical Theory of Currents*

An arbitrary field theory is described by a Lagrange density $\mathscr{L}$ which we take to depend on a set of independent fields ϕ and on their derivatives $\partial^\mu \phi \equiv \phi^\mu$. The canonical formalism rests on the following equal time commutators (ETC) [1].

$$i[\pi^0(t,\mathbf{x}), \phi(t,\mathbf{y})] = \delta(\mathbf{x}-\mathbf{y}) \ ,$$

$$i[\pi^0(t,\mathbf{x}), \pi^0(t,\mathbf{y})] = i[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})] = 0 \ . \tag{2.1}$$

Here π^0 is the time component of the canonical 4-momentum.

$$\pi^\mu = \frac{\delta \mathscr{L}}{\delta \phi_\mu} \ . \tag{2.2}$$

The Euler-Langrange equation of the theory is

$$\partial_\mu \pi^\mu = \frac{\delta \mathscr{L}}{\delta \phi} \ . \tag{2.3}$$

Consider now an infinitesmal transformation which changes $\phi(x)$ to $\phi(x) + \delta\phi(x)$. The explicit form for $\delta\phi(x)$ is assumed known; we have in mind a definite, though unspecified transformation. It is interesting to inquire what conditions on $\mathscr{L}$ insure this transformation to be a *symmetry* operation for the theory. This can be decided by examining what happens to $\mathscr{L}$ under the transformation.

$$\delta \mathscr{L} = \frac{\delta \mathscr{L}}{\delta \phi} \delta\phi + \frac{\delta \mathscr{L}}{\delta \phi^\mu} \delta\phi^\mu = \frac{\delta \mathscr{L}}{\delta \phi} \delta\phi + \pi_\mu \partial^\mu \delta\phi \ . \tag{2.4}$$

If *without the use of equations of motion* we can show that $\delta\mathscr{L}$ is a total divergence of some object Λ^μ,

$$\delta\mathscr{L} = \frac{\delta\mathscr{L}}{\delta\phi}\,\delta\phi + \pi_\mu\partial^\mu\delta\phi = \partial_\mu\Lambda^\mu \;, \tag{2.5}$$

then the action, $I = \int d^4x\mathscr{L}$, is not affected by the transformation, and the transformation is a symmetry operation of the theory. The conserved current can now be constructed in the following fashion. *With the help of the equations of motion* (2.3), an alternate formula for $\delta\mathscr{L}$ can be given which is always true, regardless whether or not we are dealing with a symmetry operation. We have from (2.3) and (2.4),

$$\delta\mathscr{L} = \partial_\mu\pi^\mu\delta\phi + \pi^\mu\partial_\mu\delta\phi = \partial_\mu(\pi^\mu\delta\phi) \;. \tag{2.6a}$$

Equating this with (2.5) yields

$$0 = \partial_\mu[\pi^\mu\delta\phi - \Lambda^\mu] \;. \tag{2.6b}$$

Hence the conserved current is

$$J_\mu = \pi_\mu\delta\phi - \Lambda_\mu \;. \tag{2.7}$$

Two situations are now distinguished. If $\Lambda^\mu = 0$, we say that we are dealing with an *internal symmetry*; otherwise we speak of a *space-time* symmetry [2]. Examples of the former are the SU(3) x SU(3) currents of Gell-Mann.

$$\delta^a\phi = T^a\phi \;. \tag{2.8}$$

T^a is a representation matrix of the group; it is assumed that the fields transform under a definite representation. The internal group index a labels the different matrices. The internal symmetry current is

$$J_\mu{}^a = \pi_\mu T^a\phi \;. \tag{2.9}$$

Space-time symmetries are exemplified by translations.

$$\delta^\alpha\phi = \partial^\alpha\phi \;,$$

$$\delta^\alpha\mathscr{L} = \partial^\alpha\mathscr{L} \;, \tag{2.10a}$$

$$\Lambda_\mu^\alpha = g_\mu^\alpha \mathscr{L} \quad . \tag{2.10b}$$

Now the transformations are labeled by the space-time index α. The conserved quantity is the canonical energy-momentum tensor $\theta_c{}^{\mu\alpha}$.

$$\theta_c{}^{\mu\alpha} = \pi^\mu \phi^\alpha - g^{\mu\alpha}\mathscr{L} \quad . \tag{2.11}$$

In the subsequent we shall reserve the symbol J_μ and the term "current" for *internal* symmetries.

It is clear that when the internal transformation (2.8) is not a symmetry operation, i.e. $\delta\mathscr{L} \neq 0$, it is still possible to define the current (2.9), which is not conserved. By virtue of the canonical formalism, the charge density satisfies a model-independent ETC, regardless whether or not the current is conserved [3].

$$\begin{aligned} &[J_0{}^a(t, \mathbf{x}), J_0{}^b(t, \mathbf{y})] \\ &\quad = [\pi_0(t, \mathbf{x})\, T^a \phi(t, \mathbf{x}),\ \pi_0(t, \mathbf{y})\, T^b \phi(t, \mathbf{y})] \\ &\quad = i\, \pi_0(t, \mathbf{x})\, [T^a, T^b]\, \phi(t, \mathbf{x})\, \delta(\mathbf{x} - \mathbf{y}) \\ &\quad = -f_{abc}\, J_0{}^c(t, \mathbf{x})\, \delta(\mathbf{x} - \mathbf{y}) \ . \end{aligned} \tag{2.12}$$

We have used the group property of the representation matrices

$$[T^a, T^b] = i f_{abc}\, T^c \ . \tag{2.13}$$

Similarly the charge,

$$Q^a(t) = \int d^3x\, J_0{}^a(t, \mathbf{x}) \ , \tag{2.14}$$

which for conserved currents is a time independent Lorentz scalar, generates the proper transformation on the fields, even in the non-conserved case.

$$\begin{aligned} &i\,[Q^a(t), \phi(t, \mathbf{x})] \\ &\quad = i \int d^3y\, [\pi_0(t, \mathbf{y})\, T^a \phi(t, \mathbf{y}),\ \phi(t, \mathbf{x})] \\ &\quad = T^a \phi(x) = \delta^a \phi(x) \ . \end{aligned} \tag{2.15}$$

It should be remarked here that although conserved and non-conserved internal symmetry currents and charges satisfy the ETC (2.12) and (2.15), the space-time currents do not, in general, satisfy commutation relations which are insensitive to conservation, or lack thereof of the appropriate quantity; see Exercise 2.5.

The importance of relations (2.12) and (2.15) is that they have been derived without reference to the specific form of $\mathscr{L}$; i.e., without any commitment of dynamics. Thus it appears that they are *always* valid, and that any consequence that can be derived from (2.12) and (2.15) will necessarily be true. But it must be remembered that the Eqs. (2.12) and (2.15) have been obtained in a very formal way; all the difficulties of local quantum theory have been ignored. Thus we have not worried about multiplying together two operators at the same space-time point, as in (2.9); nor have we inquired whether or not the equal time limit of an unequal time commutator really exists as in (2.1), (2.12) and (2.15). It will eventually be seen that the failure of current algebra predictions can be traced to precisely these problems.

A word about non-conserved currents. It turns out that in applications of the algebra of non-conserved currents, it is necessary to make assumptions about the divergence of the current. The assumption that is most frequently made is that $\partial^\mu J_\mu$ is a "gentle" operator, though the precise definition of "gentle" depends on the context. We shall spell out in detail what we mean by "gentle"; however for the moment the following concept of "gentleness" will delimit the non-conserved currents which we shall consider. The dimension of a current, in mass units, is 3. This follows from the fact that the charge, which is dimensionless, is a space integral of a current component. Therefore $\partial^\mu J_\mu$ has dimension 4. However if the dynamics of the theory is such that all *operators* which occur in $\partial^\mu J_\mu$ carry dimension less than 4, then we say that $\partial^\mu J_\mu$ is "partially conserved".

As an explicit example, consider the axial current constructed from fermion fields, $J_5{}^\mu = i\bar{\psi}\gamma^\mu\gamma^5\psi$; and assume that the fermions satisfy the equation of motion

$$i\gamma^\mu \partial_\mu \psi = -m\psi + e\gamma^\mu A_\mu + g\phi\gamma^5\psi \ .$$

Here A^μ and ϕ are vector and pseudo-scalar boson fields respectively. Recall that the dimension of a fermion field is $\frac{3}{2}$ while that of a boson field is 1. (This is seen from the Lagrangian, which necessarily has dimension 4, so that the action, $I = \int d^4x\,\mathscr{L}$, is dimensionless. The fermion Lagrangian contains $i\bar{\psi}\gamma^\mu\partial_\mu\psi$; the derivative carries one unit of dimension, this leaves

3 for $\bar{\psi}\psi$, hence ψ has dimension $\frac{3}{2}$. The boson Lagrangian contains $\partial^\mu\phi\partial_\mu\phi$; the two derivatives use up 2 units of dimension; hence ϕ has dimension 1.) Evidently $J_5{}^\mu$ possesses in this model the divergence $\partial_\mu J_5{}^\mu = 2m\bar{\psi}\gamma^5\psi + 2g\bar{\psi}\psi\phi$. The operator $\bar{\psi}\gamma^5\psi$ has dimension 3, while $\bar{\psi}\psi\phi$ has dimension 4. Hence we say that $J_5{}^\mu$ is partially conserved only in the absence of the pseudo-scalar coupling.

Although model independent commutators for current components have been derived from canonical transformation theory, the use of these results for physical predictions requires a tacit dynamical assumption which we must expose here. The point is that in the context of transformation theory it is always possible to add to the canonical current a divergence of an anti-symmetric tensor.

$$J^\mu \to J^\mu + \partial_\lambda X^{\lambda\mu} \;,$$

$$X^{\lambda\mu} = -X^{\mu\lambda} \;. \tag{2.16}$$

Such additions, called "super-potentials", do not change the charges nor the divergence properties of the current. (Conservation of $\partial_\lambda X^{\lambda\mu}$ is assured by the anti-symmetry of the super-potentials. The fact that the super-potential does not contribute to the charges is seen as follows: $\int d^3x\, \partial_\lambda X^{\lambda 0} = \int d^3x\, \partial_i X^{i0} = 0$.) It may be that the modified current possesses a physical significance, greater than that of the canonical expression. Indeed this state of affairs occurs with the energy momentum tensor. For reasons which I shall discuss presently, the canonical expression (2.11), is usually replaced in physical discussions by the symmetric, Belinfante form; see Exercise 2.3.

$$\theta_B{}^{\mu\alpha} = \theta_c{}^{\mu\alpha} + \frac{1}{2}\partial_\lambda X^{\lambda\mu\alpha} \;, \tag{2.17a}$$

$$X^{\lambda\mu\alpha} = -X^{\mu\lambda\alpha} = \pi^\lambda\Sigma^{\mu\alpha}\phi - \pi^\mu\Sigma^{\lambda\alpha}\phi - \pi^\alpha\Sigma^{\lambda\mu}\phi \;. \tag{2.17b}$$

Here $\Sigma^{\alpha\beta}$ is the spin matrix appropriate to the field ϕ.

The modified expressions for currents will in general possess commutators which differ from the canonical ones given above (2.12). Thus our insistence on the canonical commutators, rather than some others, requires an assumption that the canonical currents have a unique physical significance. This significance can be derived from the seemingly well established fact that the electromagnetic and weak interactions are governed by the *canonical* electromagnetic and SU(3) x SU(3) currents respectively. The physical significance of the Belinfante tensor follows from the belief that gravitational interactions are described by Einstein's general relativity. In that theory gravitons couple to $\theta_B{}^{\mu\nu}$, and not to $\theta_c{}^{\mu\nu}$. (In our discussion of scale transformations, Section 7, we shall argue that a new improved energy-momentum tensor should be introduced; and correspondingly gravity theory should be modified.) It is possible to develop a general formalism based directly on the dynamical role of currents. In this context one can derive current commutators without reference to canonical transformation theory. The results are of course the same, and we shall not discuss this approach here [4].

We conclude this section by recording another commutator which can be established by canonical reasoning; see Exercise 2.4.

$$i\,[\theta^{00}(t,\mathbf{x}),\, J_0{}^a(t,\mathbf{y})] \;=\; \partial^\mu J_\mu{}^a(x)\,\delta(\mathbf{x}-\mathbf{y}) \;+\; J_i^a(x)\,\partial^i\,\delta(\mathbf{x}-\mathbf{y}) \;. \tag{2.18}$$

This has the important consequence that the divergence of a current can be expressed as a commutator.

$$i\,[\theta^{00}(t,\mathbf{x}),\, Q^a(t)] \;=\; \partial^\mu J_\mu{}^a(x) \;. \tag{2.19}$$

Formulas (2.18) and (2.19) are insensitive to the choice of θ^{00}; both the canonical and the Belinfante tensor lead to the same result. Other ETC between selected componets of $\theta^{\alpha\beta}$ and $J_\mu{}^a$ can also be derived, in a model independent fashion. We do not pursue this topic here; one can read about it in the literature [5].

2.2 Space-Time Constraints on Commutators

Although interesting physical results can be obtained from the charge-density algebra, (2.12), the applications that we shall study require commutation relations between other components of the currents. These cannot be derived canonically in a model independent form. For example, $J_k{}^a$ involves π_k, see (2.9); but the dependence of π_k on the canonical variables π^0 and ϕ is not known in general, and one cannot compute commutators involving π_k in an abstract fashion.

It is possible to determine the $[J_a{}^0, J_b{}^k]$ ETC by investigating the space-time constraints which follow from the fact (2.12) is supposed to hold in all Lorentz frames. As an example, consider the once integrated version of (2.12) for the case of conserved currents.

$$[Q^a,\, J_0{}^b(0)] \;=\; -f_{abc}\, J_0{}^c(0) \;. \tag{2.20}$$

An infinitesimal Lorentz transformation can be effected on (2.20) by commuting both sides with M^{0i}, the generator of these transformations.

$$[M^{0i}, [Q^a, J_0{}^b(0)]]$$

$$= [[M^{0i}, Q^a], J_0{}^b(0)] + [Q^a, [M^{0i}, J_0{}^b(0)]] ,$$

$$= -f_{abc} [M^{0i}, J_0{}^c(0)] . \tag{2.21a}$$

The second equality in (2.21a) follows from the first by use of the Jacobi identity. All the commutators with M^{0i} may be evaluated, since the commutator of $M^{\alpha\beta}$ with $J_\mu{}^a$ is known from the fact that the current transforms as a vector.

$$i [M^{\alpha\beta}, J_\mu{}^a(x)]$$

$$= (x^\alpha \partial^\beta - x^\beta \partial^\alpha) J_\mu{}^a(x) + (g_\mu^\alpha g^{\beta\nu} - g^{\alpha\nu} g_\mu^\beta) J_\nu{}^a(x) . \tag{2.21b}$$

It now follows that (remember the current is assumed conserved)

$$[Q^a, J_i^b(0)] = -f_{abc} J_i^c(0) . \tag{2.22a}$$

The local version of (2.22a) is

$$[J_0{}^a(t, \mathbf{x}), J_i^b(t, \mathbf{y})]$$

$$= -f_{abc} J_i^c(x) \delta(\mathbf{x} - \mathbf{y}) + S_{ij}^{ab}(y) \partial^j \delta(\mathbf{x} - \mathbf{y}) + \ldots . \tag{2.22b}$$

In (2.22b) we have inserted a gradient of a δ function; the dots indicate the possible higher derivatives of δ functions which may be present. Of course, all these gradients must disappear upon integration over $\mathbf{x}$, so that (2.22a) is regained. Such gradient terms in the ETC are called Schwinger terms (ST) [6].

Further constraints can be obtained by commuting the *local* commutator (2.12) with P^0 and M^{0i}. However, the strongest results are arrived at by commuting (2.12) with θ^{00}, rather than with once integrated moments of θ^{00} which is what P^0 and M^{0i} are. ($P^0 = \int d^3x\, \theta^{00}(0, \mathbf{x})$; $M^{0i} = -\int d^3x\, x^i\, \theta^{00}(0, \mathbf{x})$.) Thus we are led to consider

$$i\,[\theta^{00}(0, \mathbf{z}), [J_0{}^a(0, \mathbf{x}), J_0{}^b(0, \mathbf{y})]]$$

$$= -f_{abc}\, \delta(\mathbf{x}-\mathbf{y})\, i\,[\theta^{00}(0, \mathbf{z}), J_0{}^c(0, \mathbf{x})] \ . \tag{2.23a}$$

The left-hand side is rewritten in terms of the Jacobi identity; then (2.18) is used to evaluate the $[\theta^{00}, J_0{}^a]$ ETC. The result, for conserved currents, is

$$[J_0{}^b(0, \mathbf{y}), J_k^a(0, \mathbf{z})]\ \partial^k\, \delta(\mathbf{x}-\mathbf{z}) + [J_0{}^a(0, \mathbf{x}), J_k^b(0, \mathbf{z})]\ \partial^k\, \delta(\mathbf{z}-\mathbf{y})$$

$$= -f_{abc}\, \delta(\mathbf{x}-\mathbf{y})\, J_k^c(0, \mathbf{z})\, \partial^k\, \delta(\mathbf{z}-\mathbf{y}) \ . \tag{2.23b}$$

The most general form for the $[J_0{}^a, J_i^b]$ ETC consistent with the constraint (2.23b) is (see Exercise 2.6)

$$[J_0{}^a(0, \mathbf{x}), J_i^b(0, \mathbf{y})]$$

$$= -f_{abc}\, J_i^c(0, \mathbf{x})\, \delta(\mathbf{x}-\mathbf{y}) + S_{ij}^{ab}(0, \mathbf{y})\, \partial^j\, \delta(\mathbf{x}-\mathbf{y}) \ , \tag{2.24a}$$

$$S_{ij}^{ab}(0, \mathbf{y}) = S_{ji}^{ba}(0, \mathbf{y}) \ . \tag{2.24b}$$

Thus we have determined the $[J_0{}^a, J_i^b]$ ETC up to *one* derivative of the δ function; all higher derivatives should vanish. The surviving ST possess the symmetry (2.24b). It will be shown later that the ST cannot vanish. The same conclusions can be obtained when the current is partially conserved, as long as the divergence of the current is sufficiently gentle so that no ST is produced when it is commuted with $J_a{}^0$.

The above methods can be used to obtain additional constraints on various current commutators. One exploits the Jacobi identity, and the model independent commutators between selected components of $\theta^{\alpha\beta}$ and J^μ. We do not present these results here, since they are only of limited interest. However one result is sufficiently elegant to deserve explicit mention. If $S_{ij}^{ab} = \delta_{ij} S^{ab}$, where S_{ab} is a Lorentz scalar, then the $[J_i^a, J_j^b]$ ETC does not have any derivatives of δ functions [7].

2.3 *Space-Time Constraints on Green's Functions*

We must also discuss the space-time structure of Green's functions and Ward identities. The reason for emphasizing this topic here is that the theorems of current algebra concern themselves with Green's functions: scattering amplitudes, decay amplitudes and the like; while the most felicitous way of obtaining these results is by the use of Ward identities.

Consider the T product of two operators A and B.

$$\mathrm{T}(x) = \mathrm{T}A(x)B(0)\ ,$$

$$= \theta(x_0)A(x)B(0) + \theta(-x_0)B(0)A(x)\ . \tag{2.25}$$

Matrix elements of $\mathrm{T}(x)$ are related to Green's functions. However, a Green's function must be Lorentz covariant, while $\mathrm{T}(x)$ need not have this property because of the time ordering. It is necessary, in the general case, to add to $\mathrm{T}(x)$ another non-covariant term, called a seagull, $\tau(x)$, so that the sum is covariant. The sum of a time ordered product with the covariantizing seagull is called a T* product.

$$\mathrm{T}^*(x) = \mathrm{T}(x) + \tau(x)\ . \tag{2.26}$$

It is required that $\mathrm{T}^*(x)$ and $\mathrm{T}(x)$ coincide for $x_0 \neq 0$; hence $\tau(x)$ has support only at $x_0 = 0$; i.e., $\tau(x)$ will involve δ functions of x_0 and derivatives thereof.

We now investigate under what conditions $\mathrm{T}(x)$ is not covariant. We also show how to construct the covariantizing seagull. Finally we examine under what conditions Feynman's conjecture concerning the cancellation of Schwinger terms against divergences of seagulls is valid. (Feynman's conjecture will be explained, when we come to it.) To effect this analysis it is necessary to assume that the $[A, B]$ ETC is known.

$$[A(0,\mathbf{x}), B(0)] = C(0)\delta(\mathbf{x}) + S^i(0)\,\partial_i\,\delta(\mathbf{x})\ . \tag{2.27}$$

In offering (2.27) we have assumed, for simplicity, one ST; higher derivatives can easily be accommodated by the present technique.

Our analysis [8] makes use of the device of writing non-covariant expressions in a manifestly covariant, but frame dependent notation. A unit