behavior of amplitudes *is not*, in general, correctly given by this technique has recently been described as " the breakdown of the BJL theorem".

Note that the BJL theorem defines the commutator from the T product, rather than from the covariant T* product. However, in perturbation theory one calculates only the covariant object. Hence the T product must be separated. This is achieved by remembering that the difference btween T and T* is local in position space, hence it is a polynomial of q_0 in momentum space. Therefore, before applying the BJL technique to the expressions calculated in perturbation theory, all polynomials in q_0 must be dropped.

It is clear that the expansion in inverse powers of q_0 can be extended beyond the first. From (3.3a), it is easy to see that if the $[A, B]$ ETC vanishes, then we have

$$\lim_{q_0 \to \infty} q_0^2 \, \mathrm{T}(q) = -\int d^3x \, e^{-i\mathbf{q}\cdot\mathbf{x}} \langle \alpha | [\dot{A}(0,\mathbf{x}), B(0)] | \beta \rangle \, . \qquad (3.12)$$

Again, if this limit is divergent, then this matrix element of the $[\dot{A}, B]$ ETC is infinite. Eventually commutators with sufficient number of time derivatives probably are infinite, since it is unlikely that the expansion in inverse powers of q_0 can be extended without limit.

References

[1] J. D. Bjorken, Phys. Rev. 148 (1966) 1467.

[2] K. Johnson and F. E. Low, Prog. Theor. Phys. (Kyoto), Supp. 37–38, (1966) 74.

[3] This method was developed in conservations with Prof. I. Gerstein.

4. The $\pi^0 \to 2\gamma$ Problem

4.1 Preliminaries

The neutral pion is observed to decay into 2 photons with a width of the order of 10 eV. This experiment measures the matrix element $M(p,q) = \langle \pi, k | \gamma, p; \gamma', q \rangle$; p and q are the 4-momenta of the photons, $k = p + q$ is the 4-momentum of the pion. $M(p,q)$ has the form

$$\epsilon_\mu(p)\,\epsilon'_\nu(q)\,\mathrm{T}^{\mu\nu}(p,q) \; ,$$

i.e., $T^{\mu\nu}(p, q)$ is the previous matrix element with the photon polarization vectors $\epsilon_\mu(p)\epsilon'_\nu(q)$ removed. The tensor $T^{\mu\nu}$ has the following structure.

$$T^{\mu\nu}(p, q) = \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta T(k^2) \ . \tag{4.1}$$

This is dictated by Lorentz covariance and parity conservation (the pion is a pseudoscalar, $T^{\mu\nu}$ must be a pseudo-tensor, hence the factor $\epsilon^{\mu\nu\alpha\beta}$). Gauge invariance ($p_\mu T^{\mu\nu}(p, q) = 0 = q_\nu T^{\mu\nu}(p, q)$) and Bose symmetry ($T^{\mu\nu}(p, q) = T^{\nu\mu}(q, p)$) are seen to hold.

We shall keep q^2 and p^2, the photon variables, on their mass shell $q^2 = p^2 = 0$. The pion variable k^2 is, of course, equal to the pion mass squared μ^2, but for our arguments we allow it to vary away from this point. This continuation off the mass shell may be effected by the usual LSZ method.

$$\begin{aligned} T^{\mu\nu}(p, q) &= \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta T(k^2) \\ &= (\mu^2 - k^2) \langle 0 | \phi(0) | \gamma, p; \gamma', q \rangle \ . \end{aligned} \tag{4.2}$$

Here ϕ is an interpolating field for the pion. We, of course, do not assert that it is *the* pion field — such an object may not exist. It merely is some local operator which has a non-vanishing matrix element between the vacuum and the single pion state, normalized to unity $\langle 0 | \phi(0) | \pi \rangle = 1$.

4.2 Sutherland-Veltman Theorem

Following Sutherland and Veltman [1], we now prove that if the divergence of the axial current is used as the pion interpolating field, then $T(0) = 0$, as long as the conventional current algebraic ideas are valid. This is a *mathematical* fact, without direct experimental content. However, since μ^2 is small, compared to all other mass parameters relevant to this problem, one may expect that $T(\mu^2) \approx T(0)$. This smoothness hypothesis is based on the supposition that the divergence of the axial current is a "gentle" operator whose matrix elements do not have any dynamically unnecessary rapid variation. This is the content of PCAC, which is very successful notion in other contexts. Unfortunately, in the present application, one cannot understand the *experimental* fact that $T(\mu^2) \neq 0$.

After Sutherland and Veltman pointed out this *experimental* failure of PCAC, the most widely accepted explanation was that $T(k^2)$ *was* rapidly varying, for unknown reasons. This is not impossible, since it has happened before in current algebra-PCAC applications that a source of rapid variation for a particular amplitude was at first overlooked. However, in the present instance, as the years passed by, no reason was forthcoming to explain the putative rapid variation.

The Sutherland-Veltman argument begins by representing the off mass shell pion amplitude (4.2) by

$$T^{\mu\nu}(p,q) = e^2(\mu^2 - k^2)\int d^4x\, d^4y\, e^{-ip\cdot x}\, e^{-iq\cdot y} \times \langle 0|T^* J^\mu(x) J^\nu(y)\,\phi(0)|0\rangle\ . \tag{4.3}$$

Here J^μ is the electromagnetic current. The pion field is replaced by the divergence of the neutral axial current $J_5{}^\alpha$.

$$\phi(0) = \frac{\partial_\alpha J_5{}^\alpha(0)}{F\mu^2}\ . \tag{4.4}$$

$F\mu^2$ is the appropriate factor which assures the proper normalization for the pion field, defined by (4.4).

$$\langle 0|J_5{}^\alpha(0)|\pi\rangle \equiv ip^\alpha F\ ,$$

$$\langle 0|\partial_\alpha J_5{}^\alpha(0)|\pi\rangle = \mu^2 F\ . \tag{4.5}$$

Thus

$$\begin{aligned} T^{\mu\nu}(p,q) &= \frac{e^2(\mu^2-k^2)}{F\mu^2}\int d^4x\, d^4y\, e^{-ip\cdot x}\, e^{-iq\cdot y} \times \langle 0|T^* J^\mu(x) J^\nu(y)\,\partial_\alpha J_5{}^\alpha(0)|0\rangle\ ,\\ &= \frac{e^2(\mu^2-k^2)}{F\mu^2}\int d^4x\, d^4y\, e^{-ip\cdot x}\, e^{-iq\cdot y} \times \partial_\alpha \langle 0|T^* J^\mu(x) J^\nu(y) J_5{}^\alpha(0)|0\rangle\ ,\\ &= \frac{(\mu^2-k^2)}{F\mu^2}\, k_\alpha T^{\alpha\mu\nu}(p,q)\ ; \end{aligned} \tag{4.6a}$$

where $T^{\alpha\mu\nu}(p,q)$ is defined by

$$T^{\alpha\mu\nu}(p,q) = -ie^2\int d^4x\, d^4y\, e^{-ip\cdot x}\, e^{-iq\cdot y} \times \langle 0|T^* J^\mu(x) J^\nu(y) J_5{}^\alpha(0)|0\rangle \,. \tag{4.6b}$$

The justification for passing from the first to the second term on the right-hand side of (4.6a) is the current algebra satisfied by J_5^0 and J^μ: apart from possible ST the currents commute.

$$[J_5^0(0,\mathbf{x}), J^\mu(0)] = 0 + \text{ST} \,. \tag{4.7}$$

The ST is handled by one of three ways. One may simply assume that it is absent; since the ETC does not involve two identical currents, there is no proof that a ST must be present. Alternatively a weaker assumption is that the ST is a c number. It is easy to see that since the vacuum expectation value of a current vanishes, a c-number ST would not interfere with passing the derivative through the T* product. Finally the weakest assumption that one can make is Feynman's conjecture — without discussing the nature of any possible ST, it is asserted that the naive procedure is the correct one, due to cancellation of ST with divergences of seagulls.

The tensor $T^{\alpha\mu\nu}(p,q)$ must possess odd parity, because $J_5{}^\alpha$ is a pseudovector; it must satisfy the Bose symmetry: $T^{\alpha\mu\nu}(p,q) = T^{\alpha\nu\mu}(q,p)$; finally it must be transverse to p_μ and q_ν: $p_\mu T^{\alpha\mu\nu}(p,q) = 0$, $q_\nu T^{\alpha\mu\nu}(p,q) = 0$. The last condition follows from the conservation of J^μ and the current algebra satisfied by J^0 with J^μ and $J_5{}^\alpha$. Again all these commutators vanish apart from possible ST; the latter being ignored in this calculation.

$$[J^0(0,\mathbf{x}), J^\mu(0)] = 0 + \text{ST} \,, \tag{4.8a}$$

$$[J^0(0,\mathbf{x}), J_5{}^\mu(0)] = 0 + \text{ST} \,. \tag{4.8b}$$

The following form for $T^{\alpha\mu\nu}(p,q)$ is the most general structure, free from kinematical singularities, satisfying the above requirements (remember that $p^2 = q^2 = 0$).

$$\begin{aligned} T^{\alpha\mu\nu}(p,q) = {} & \epsilon^{\mu\nu\omega\phi} p_\omega q_\phi k^\alpha F_1(k^2) \\ & + (\epsilon^{\alpha\mu\omega\phi} q^\nu - \epsilon^{\alpha\nu\omega\phi} p^\mu) p_\omega q_\phi F_2(k^2) \\ & + (\epsilon^{\alpha\mu\omega\phi} p^\nu - \epsilon^{\alpha\nu\omega\phi} q^\mu) p_\omega q_\phi F_3(k^2) \\ & + \epsilon^{\alpha\mu\nu\omega}(p_\omega - q_\omega) k^2 F_3(k^2)/2 \,. \end{aligned} \tag{4.9}$$

It now follows that

$$k_\alpha \mathrm{T}^{\alpha\mu\nu}(p,q) = \epsilon^{\mu\nu\omega\phi} p_\omega q_\phi k^2 [F_1(k^2) - F_3(k^2)] \ . \qquad (4.10a)$$

Comparison with (4.6a) and (4.1) finally gives

$$\mathrm{T}(k^2) = \frac{(\mu^2 - k^2)}{F\mu^2} k^2 [F_1(k^2) - F_3(k^2)] \ . \qquad (4.10b)$$

As we mentioned, the F_i are free from kinematical singularities; since we are working to lowest order in electromagnetism, they do not possess dynamical singularities at $k^2 = 0$. Hence we find, as promised, $\mathrm{T}(0) = 0$. Note that PCAC is not used to obtain the mathematical statement $\mathrm{T}(0) = 0$. This hypothesis becomes necessary only when T(0) is related to $\mathrm{T}(\mu^2)$. It will now be shown that even the mathematical prediction is invalid in the σ model.

4.3 *Model Calculation*

We calculate [2] the off mass shell pion decay constant, $\mathrm{T}(k^2)$, in the σ model where all the assumptions of the Sutherland-Veltman theorem seem to be satisfied [3]. The Lagrangian is

$$\begin{aligned} \mathscr{L} = {} & \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}\mu^2\phi^2 + \tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma \\ & - \tfrac{1}{2}(\mu^2 + 2\lambda F^2)\sigma^2 + e\bar{\psi}\gamma^\mu\psi A_\mu + g\bar{\psi}(\sigma + \phi\gamma_5)\psi \\ & - \lambda[(\phi^2 + \sigma^2)^2 - 2F\sigma(\sigma^2 + \phi^2)] \ . \end{aligned} \qquad (4.11)$$

Here ψ, ϕ and σ are fields for the "proton", "pion" and σ particle, each possessing the respective masses m, μ and $(\mu^2 + 2\lambda F^2)^{1/2}$. The proton interacts with the pion and σ in a chirally symmetric fashion with strength g. The proton also has an electromagnetic interaction; since we work to lowest order in that interaction, it is sufficient to consider the electromagnetic potential A^μ as an external perturbation. There are also meson self-couplings with strength λ, which are necessary for the consistency of the model, but which do not affect the present discussion. The parameter F is equal to $2mg^{-1}$. All isospin effects are ignored, since they are irrelevant to the argument.

The model possesses a neutral axial current J_5^{α} whose divergence according to the equations of motion of the theory is the pion field.

$$J_5^{\alpha} = i\bar{\psi}\gamma^{\alpha}\gamma^5\psi + 2(\sigma\partial^{\alpha}\phi - \phi\partial^{\alpha}\sigma) - F\partial^{\alpha}\phi , \tag{4.12a}$$

$$\partial_{\alpha}J_5^{\alpha} = \mu^2 F\phi . \tag{4.12b}$$

The electromagnetic current $J^{\mu} = \bar{\psi}\gamma^{\mu}\psi$ and the axial current satisfy conventional current commutators. Of course, no ST is present canonically, in the time-component algebra so we cannot ascertain whether or not Feynman's conjecture is satisfied.

In this theory the pion can decay into two photons by dissociating first into a virtual proton-antiproton pair, which then emits two photons. The lowest order graphs are those of Fig. 4-1. These have the integral representation

$$\mathrm{T}^{\mu\nu}(p,q) = \Gamma^{\mu\nu}(p,q) + \Gamma^{\nu\mu}(q,p) ,$$

$$\Gamma^{\mu\nu}(p,q) = ige^2\int\frac{\mathrm{d}^4 r}{(2\pi)^4}\,\mathrm{Tr}\,\gamma^5[\gamma_{\alpha}r^{\alpha} + \gamma_{\alpha}p^{\alpha} - m]^{-1}$$

$$\times\ \gamma^{\mu}[\gamma_{\alpha}r^{\alpha} - m]^{-1}\gamma^{\nu}[\gamma_{\alpha}r^{\alpha} - \gamma_{\alpha}q^{\alpha} - m]^{-1} . \tag{4.13a}$$

The integral appears to diverge linearly; however, after the trace is performed one is left with a finite expression.

$$\Gamma^{\mu\nu}(p,q) = 4mige^2\epsilon^{\mu\nu\alpha\beta}p_{\alpha}q_{\beta}\int\frac{\mathrm{d}^4 r}{(2\pi)^4}$$

$$\times\ [(r+p)^2 - m^2]^{-1}[r^2 - m^2]^{-1}[(r-q)^2 - m^2]^{-1} . \tag{4.13b}$$

The remaining evaluation is elementary [4]. The answer is

$$\Gamma^{\mu\nu}(p,q) = \frac{mge^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}p_{\alpha}q_{\beta}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,[m^2 - k^2xy]^{-1} . \tag{4.13c}$$

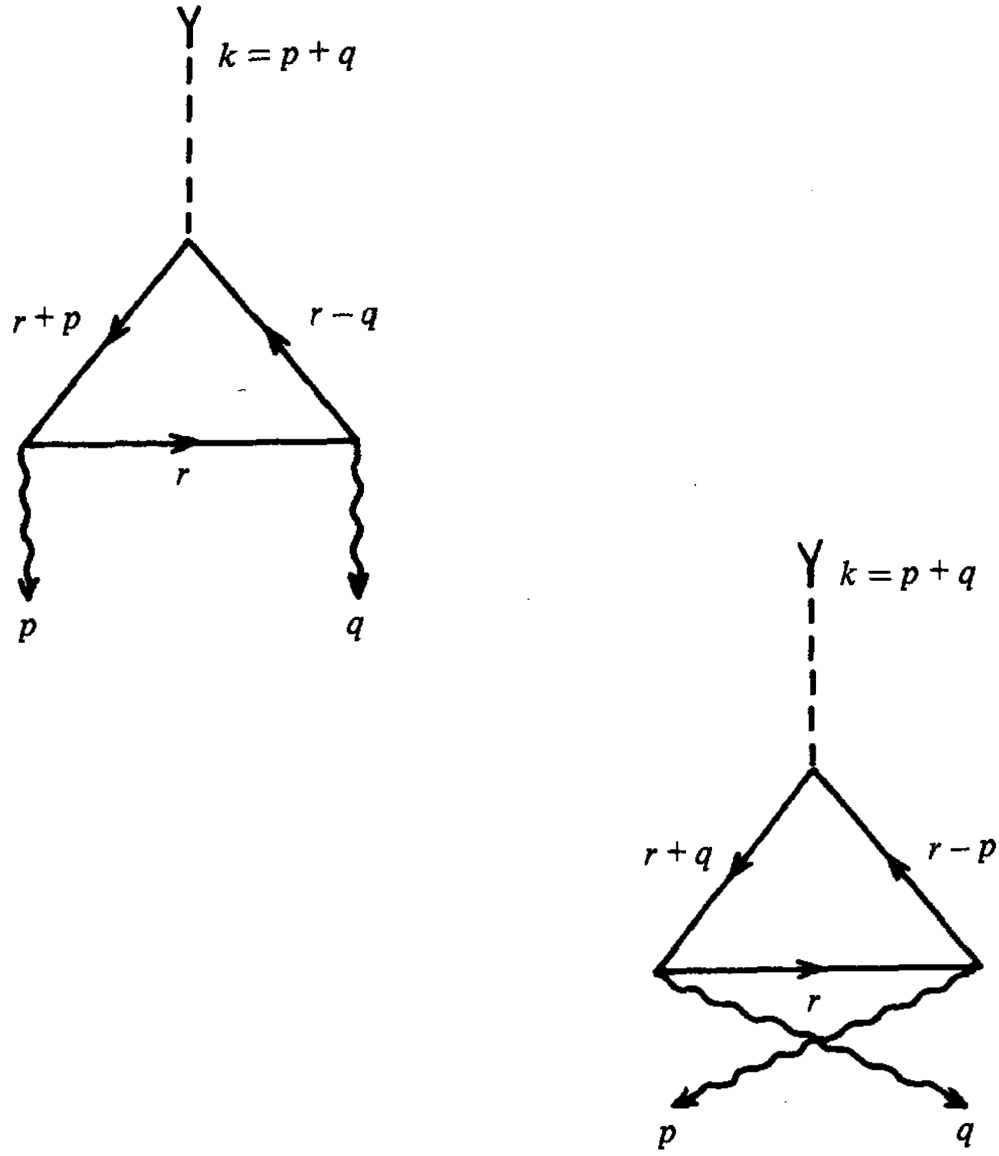

Fig. 4-1. Feynman graphs describing the $\pi^0 \to 2\gamma$ amplitude, in lowest order for the theory given by Eq. (4.11).

In the notation (4.1) we find

$$T(k^2) = \frac{mge^2}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy \, [m^2 - k^2 xy]^{-1} \; ,$$

$$T(0) = \frac{ge^2}{4\pi^2 m} = \frac{e^2}{2\pi^2 F} \neq 0 \; . \qquad (4.14)$$

For future reference, note that the large m behavior of $T(k^2)$ is $ge^2/4\pi^2 m$.

Our calculation has demonstrated that the *mathematical* portion of the Sutherland-Veltman theorem is false. Since $T(k^2)$ is perfectly smooth for small k^2, $T(k^2) \approx T(0)\,[1 + k^2/(12m^2)]$, we see that the *experimental* part of that theorem is also incorrect in this model. The reason does not lie in any unexpected rapid variation, but rather in the failure of conventional current algebra.

4.4 *Anomalous Ward Identity*

To gain further understanding into the problem, we calculate the function $T^{\alpha\mu\nu}(p,q)$, (4.6b). The relevant Feynman graphs are given in Fig. 4-2. In the first two graphs, Fig. 4-2a, the axial current attaches directly to the Fermion loop. In the last two, Fig. 4-2b, it passes first through the virtual pion with the coupling $2mg^{-1}$, thus acquiring the necessary pion pole. The integral representation is

$$T^{\alpha\mu\nu}(p,q) = T_1{}^{\alpha\mu\nu}(p,q) + T_2{}^{\alpha\mu\nu}(p,q) \,, \tag{4.15a}$$

$$T_1{}^{\alpha\mu\nu}(p,q) = \Gamma^{\alpha\mu\nu}(p,q) + \Gamma^{\alpha\nu\mu}(q,p) \,, \tag{4.15b}$$

$$\Gamma^{\alpha\mu\nu}(p,q) = ie^2 \int \frac{d^4 r}{(2\pi)^4} \operatorname{Tr} \gamma^5 \gamma^\alpha [\gamma_\beta r^\beta + \gamma_\beta p^\beta - m]^{-1} \times \gamma^\mu [\gamma_\beta r^\beta - m]^{-1} \gamma^\nu [\gamma_\beta r^\beta - \gamma_\beta q^\beta - m]^{-1} \,, \tag{4.15c}$$

$$T_2{}^{\alpha\mu\nu}(p,q) = -\frac{2mg^{-1}}{k^2 - \mu^2} k^\alpha T^{\mu\nu}(p,q) \,. \tag{4.15d}$$

Evidently the verification of the axial Ward identity

$$(F\mu^2)^{-1} (\mu^2 - k^2) k_\alpha T^{\alpha\mu\nu}(p,q) = T^{\mu\nu}(p,q) \,, \tag{4.16a}$$

which was used in the derivation of the Sutherland-Veltman theorem, is equivalent to showing

$$k_\alpha T_1{}^{\alpha\mu\nu}(p,q) = 2mg^{-1} T^{\mu\nu}(p,q) \,. \tag{4.16b}$$

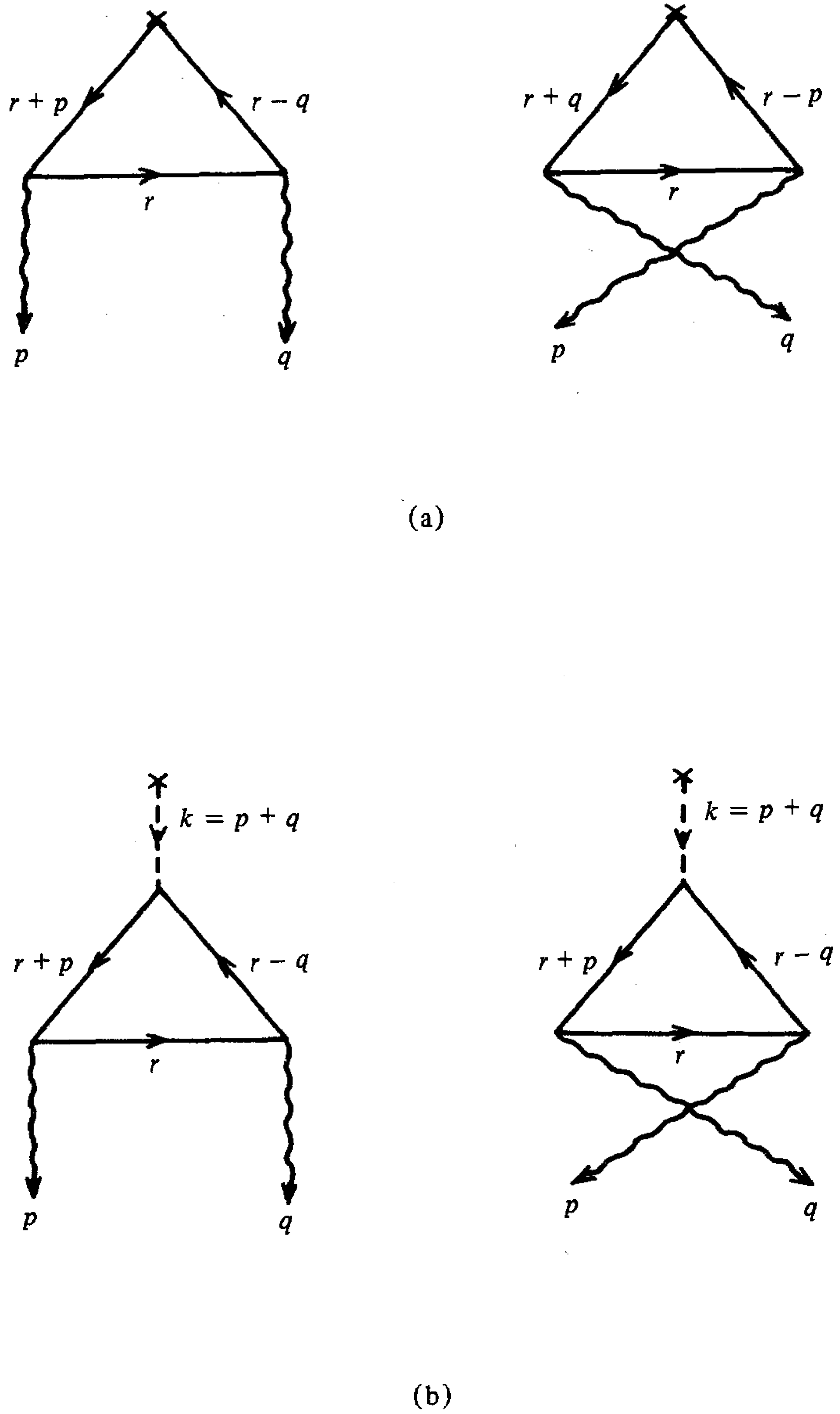

Fig. 4-2. Feynman graphs describing the $J_5^{\alpha} \to 2\gamma$ amplitude, in lowest order for the theory given by Eq. (4.11).

The vector Ward identity, i.e., gauge invariance, is also necessary for the theorem. In the present notation it is equivalent to

$$p_\mu \mathrm{T}_1^{\alpha\mu\nu}(p,q) = q_\nu \mathrm{T}_1^{\alpha\mu\nu}(p,q) = 0 \; . \tag{4.17}$$

We now demonstrate that both (4.16) and (4.17) cannot be maintained simultaneously for $\mathrm{T}_1^{\alpha\mu\nu}$ as given by (4.15).

The important property of the graphs in Fig. 4-2a which is responsible for this anomalous behavior is their linear divergence. Unlike in the evaluation of $\mathrm{T}^{\mu\nu}$, (4.13), performing the trace does not remove this divergence. In a linearly divergent integral it is illegitimate to shift the variable of integration. This is easily seen on one dimension. Consider

$$\Delta(a) = \int_{-\infty}^{\infty} \mathrm{d}x \, [f(x+a) - f(x)] \; . \tag{4.18a}$$

If one can shift the integration variable in the first integral $x+a \to x$; one can conclude that $\Delta(a) = 0$. To see that $\Delta(a)$ need not vanish, let us expand the integrand.

$$\Delta(a) = \int_{-\infty}^{\infty} \mathrm{d}x \, [a f'(x) + \frac{a^2}{2} f''(x) + \ldots] \; . \tag{4.18b}$$

Integrating by parts, we find

$$\Delta(a) = a[f(\infty) - f(-\infty)] + \frac{a^2}{2}[f'(\infty) - f'(-\infty)] + \ldots \quad . \tag{4.18c}$$

When the integral $\int_{-\infty}^{\infty} \mathrm{d}x \, f(x)$ converges (or at most diverges logarithmically) we have $0 = f(\pm\infty) = f'(\pm\infty) \ldots$, and $\Delta(a)$ vanishes. However, for a linearly divergent integral $0 \neq f(\pm\infty)$, $0 = f'(\pm\infty) \ldots$ and $\Delta(a)$ need not vanish.

$$\Delta(a) = a[f(\infty) - f(-\infty)] \; . \tag{4.18d}$$

Such a contribution is called a "surface term". This state of affairs persists in 4-dimensional, Minkowski space integrals. Consider

$$\Delta^\mu(a) = i \int \frac{\mathrm{d}^4 r}{(2\pi)^4} \left[\frac{r^\mu + a^\mu}{([r+a]^2 - m^2)^2} - \frac{r^\mu}{(r^2 - m^2)^2} \right] \; . \tag{4.19a}$$

Here a is an arbitrary 4-vector. The surface term may be evaluated. The result is non-vanishing; see Exercise 4.1.

$$\Delta^{\mu}(a) = -\frac{a^{\mu}}{32\pi^2} \quad . \tag{4.19b}$$

The consequence of this for our problem is that the integral $\Gamma^{\alpha\mu\nu}$ (4.15c), which contributes to $T_1{}^{\alpha\mu\nu}$, is not uniquely defined. The point is that in exhibiting (4.15c) we have routed the integration momentum r in a particular fashion: the fermion leg between the two photons carries r. However any other routing could also be chosen, so that the fermion leg between the photons carries the 4-momentum $r+a$, where a is an arbitrary four vector. If the integral were not linearly divergent, then a shift of integration would return this routing to the previous one; but in the present instance such shifts produce surface terms.

In conventional evaluations of divergent Feynman graphs, such ambiguities are usually ignored. Typically cut-offs are introduced, which eliminate these problems and then the cut-offs are removed by the renormalization procedure. For our purposes we need to keep track of all the possible sources of ambiguity. Thus we replace the expression for $\Gamma^{\alpha\mu\nu}$, (4.15c), by a class of expressions, parametrized by an arbitrary 4-vector a^{μ}.

$$\Gamma^{\alpha\mu\nu}(p,q\,|a) = \Gamma^{\alpha\mu\nu}(p,q) + \Delta^{\alpha\mu\nu}(p,q\,|a) \ , \tag{4.20a}$$

$$\begin{aligned}
&\Delta^{\alpha\mu\nu}(p,q\,|a) \\
&= ie^2 \int \frac{d^4 r}{(2\pi)^4} \operatorname{Tr} \gamma^5 \gamma^\alpha \; [\gamma_\beta r^\beta + \gamma_\beta a^\beta + \gamma_\beta p^\beta - m]^{-1} \\
&\quad \times \gamma^\mu [\gamma_\beta r^\beta + \gamma_\beta a^\beta - m]^{-1} \gamma^\nu [\gamma_\beta r^\beta + \gamma_\beta a^\beta - \gamma_\beta q^\beta - m]^{-1} \\
&\quad\quad - [\gamma_\beta r^\beta + \gamma_\beta p^\beta - m]^{-1} \gamma^\mu [\gamma_\beta r^\beta - m]^{-1} \\
&\quad \times \gamma^\nu [\gamma_\beta r^\beta - \gamma_\beta q^\beta - m]^{-1} \quad .
\end{aligned} \tag{4.20b}$$

The surface term is evaluated, see Exercise 4.2.

$$\Delta^{\alpha\mu\nu}(p,q\,|a) = -\frac{e^2}{8\pi^2} \epsilon^{\alpha\mu\nu\beta} a_\beta \ . \tag{4.21}$$

The arbitrariness of a_β is limited somewhat by the plausibility requirement that no vectors, other than those already present in the problem at hand, should be introduced. Hence we set $a_\beta = (a+b)p_\beta + bq_\beta$. Corresponding to the class of functions $\Gamma^{\alpha\mu\nu}(p,q\,|\,a)$ we have a class of functions $T_1{}^{\alpha\mu\nu}(p,q\,|\,a)$. Accordingly (4.15b) and (4.21)

$$T_1{}^{\alpha\mu\nu}(p,q\,|\,a) = T_1{}^{\alpha\mu\nu}(p,q) - \frac{e^2}{8\pi^2}\, a\,\epsilon^{\alpha\mu\nu\beta}(p_\beta - q_\beta)\,. \qquad (4.22)$$

Note that Bose symmetry has been maintained.

Any member of the class of functions $T_1{}^{\alpha\mu\nu}(p,q\,|\,a)$ may be considered "correct". The various functions differ among themselves only by a polynomial in p and q, i.e., by a covariant seagull. We now attempt to determine a by imposing the axial and the vector Ward identities. We are hoping that $T_1{}^{\alpha\mu\nu}(p,q\,|\,a)$, for some definite value of a, will satisfy these identities. It will be seen that no such value for a exists.

It is possible to evaluate $T_1{}^{\alpha\mu\nu}(p,q)$ as given by (4.15), and therefore to exhibit an explicit formula for $T_1{}^{\alpha\mu\nu}(p,q\,|\,a)$. The evaluation is effected by conventional methods, except it must be always remembered that shifts of integration variables produce non-vanishing, but well defined terms. A remarkable thing that occurs is that the end result is finite; symmetric integration removes the linear divergences as well as the sub-dominant logarithmic divergence. Thus a finite, unambiguous formula for $T_1{}^{\alpha\mu\nu}(p,q\,|\,a)$ may be arrived at. (The illegitimacy of shifts of integration follows from the *superficial* divergence properties of integral, even if accidentally the result is finite.) The detailed evaluation of $T_1{}^{\alpha\mu\nu}(p,q)$ has been given in the literature [5]; for our present purposes of verifying the Ward identities we do not need this formula, the integral representation (4.15) will suffice.

Consider first the axial Ward identity. We wish to learn the form of $k_\alpha T_1{}^{\alpha\mu\nu}(p,q)$. From (4.15c) it follows that

$$\begin{aligned} &k_\alpha \Gamma^{\alpha\mu\nu}(p,q) \\ &= (p_\alpha + q_\alpha)\,\Gamma^{\alpha\mu\nu} \\ &= ie^2 \int \frac{d^4 r}{(2\pi)^4}\, \mathrm{Tr}\, \gamma^5 (\gamma_\beta p^\beta + \gamma_\beta q^\beta)\, [\gamma_\beta r^\beta + \gamma_\beta p^\beta - m]^{-1} \\ &\quad \times \gamma^\mu [\gamma_\beta r^\beta - m]^{-1} \gamma^\nu [\gamma_\beta r^\beta - \gamma_\beta q^\beta - m]^{-1}\,. \end{aligned} \qquad (4.23a)$$

After rewriting $\gamma_\beta p^\beta + \gamma_\beta q^\beta$ as $2m + (\gamma_\beta p^\beta + \gamma_\beta r^\beta - m) - (\gamma_\beta r^\beta - \gamma_\beta q^\beta + m)$ we have

$$k_\alpha \Gamma^{\alpha\mu\nu}(p,q)$$

$$= (2mg^{-1}) ige^2 \int \frac{d^4 r}{(2\pi)^4} \gamma^5 [\gamma_\beta r^\beta + \gamma_\beta p^\beta - m]^{-1}$$

$$\times \gamma^\mu [\gamma_\beta r^\beta - m]^{-1} \gamma^\nu [\gamma_\beta r^\beta - \gamma_\beta q^\beta - m]^{-1}$$

$$+ ie^2 \int \frac{d^4 r}{(2\pi)^4} \mathrm{Tr}\, \gamma^5 \gamma^\mu [\gamma_\beta r^\beta - m]^{-1} \gamma^\nu [\gamma_\beta r^\beta - \gamma_\beta q^\beta - m]^{-1}$$

$$- ie^2 \int \frac{d^4 r}{(2\pi)^4} \mathrm{Tr}\, \gamma^5 [\gamma_\beta r^\beta - \gamma_\beta q^\beta + m][\gamma_\beta p^\beta + \gamma_\beta r^\beta - m]^{-1}$$

$$\times \gamma^\mu [\gamma_\beta r^\beta - m]^{-1} \gamma^\nu [\gamma_\beta r^\beta - \gamma_\beta q^\beta - m]^{-1} . \qquad (4.23b)$$

The first integral is recognized as $2mg^{-1}$ times $\Gamma^{\mu\nu}(p,q)$; see (4.13). In the third integral $\gamma_\beta r^\beta - \gamma_\beta q^\beta + m$ may be taken through the γ^5, thus changing the overall sign and the sign of m. Then the cyclicity of the trace allows one to transpose that term to the end of that expression, thus cancelling the last propagator. We are now left with

$$k_\alpha \Gamma^{\alpha\mu\nu}(p,q) = 2mg^{-1} \Gamma^{\mu\nu}(p,q)$$

$$+ ie^2 \int \frac{d^4 r}{(2\pi)^4} \mathrm{Tr}\, \gamma^5 \gamma^\mu [\gamma_\beta r^\beta - m]^{-1} \gamma^\nu [\gamma_\beta r^\beta - \gamma_\beta q^\beta - m]^{-1}$$

$$+ ie^2 \int \frac{d^4 r}{(2\pi)^4} \mathrm{Tr}\, \gamma^5 [\gamma_\beta p^\beta + \gamma_\beta r^\beta - m]^{-1} \gamma^\mu [\gamma_\beta r^\beta - m]^{-1} \gamma^\nu .$$

(4.23c)

Each of the two integrals must vanish since it is impossible to form a two-index pseudotensor which depends on only one vector. We find therefore

$$k_\alpha T_1{}^{\alpha\mu\nu}(p,q) = 2mg^{-1} T^{\mu\nu}(p,q) ,$$

$$k_\alpha T_1{}^{\alpha\mu\nu}(p,q\,|\,a) = 2mg^{-1} T^{\mu\nu}(p,q) + \frac{a e^2}{4\pi^2} \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta . \qquad (4.24)$$

The conclusion is that in order to satisfy the axial Ward identity the routing of the integration variable must be as in Fig. 4-2; i.e., a must be set to zero. Note that this verification required no shifts of integration variable. The vector Ward identity, i.e., gauge invariance cannot be established in the same fashion, as we now demonstrate.

We wish to learn the form of $p_\mu T_1{}^{\alpha\mu\nu}(p,q)$. According to (4.15)

$$
\begin{aligned}
p_\mu T_1{}^{\alpha\mu\nu}(p,q) &= p_\mu \Gamma^{\alpha\mu\nu}(p,q) + p_\mu \Gamma^{\alpha\mu\nu}(q,p) \\
&= ie^2 \int \frac{d^4 r}{(2\pi)^4} \operatorname{Tr} \gamma^5 \gamma^\alpha [\gamma_\beta r^\beta + \gamma_\beta p^\beta - m]^{-1} \\
&\quad \times \gamma_\beta p^\beta [\gamma_\beta r^\beta - m]^{-1} \gamma^\nu [\gamma_\beta r^\beta - \gamma_\beta q^\beta - m]^{-1} \\
&\quad + ie^2 \int \frac{d^4 r}{(2\pi)^4} \operatorname{Tr} \gamma^5 \gamma^\alpha [\gamma_\beta r^\beta + \gamma_\beta q^\beta - m]^{-1} \\
&\quad \times \gamma^\nu [\gamma_\beta r^\beta - m]^{-1} \gamma_\beta p^\beta [\gamma_\beta r^\beta - \gamma_\beta p^\beta - m]^{-1} .
\end{aligned}
\tag{4.25a}
$$

Use of the identities

$$
\begin{aligned}
&[\gamma_\beta r^\beta + \gamma_\beta p^\beta - m]^{-1} \gamma_\beta p^\beta [\gamma_\beta r^\beta - m]^{-1} \\
&\quad = [\gamma_\beta r^\beta - m]^{-1} - [\gamma_\beta r^\beta + \gamma_\beta p^\beta - m]^{-1} ,
\end{aligned}
$$

$$
\begin{aligned}
&[\gamma_\beta r^\beta - m]^{-1} \gamma_\beta p^\beta [\gamma_\beta r^\beta - \gamma_\beta p^\beta - m]^{-1} \\
&\quad = [\gamma_\beta r^\beta - \gamma_\beta p^\beta - m]^{-1} - [\gamma_\beta r^\beta - m]^{-1} ,
\end{aligned}
$$

allows (4.25a) to be written as

$$p_\mu T_1{}^{\alpha\mu\nu}(p,q)$$

$$= ie^2 \int \frac{d^4r}{(2\pi)^4} \operatorname{Tr} \gamma^5 \gamma^\alpha [\gamma_\beta r^\beta - m]^{-1} \gamma^\nu [\gamma_\beta r^\beta - \gamma_\beta q^\beta - m]^{-1}$$

$$- ie^2 \int \frac{d^4r}{(2\pi)^4} \operatorname{Tr} \gamma^5 \gamma^\alpha [\gamma_\beta r^\beta + \gamma_\beta p^\beta - m]^{-1} \gamma^\nu [\gamma_\beta r^\beta - \gamma_\beta q^\beta - m]^{-1}$$

$$+ ie^2 \int \frac{d^4r}{(2\pi)^4} \operatorname{Tr} \gamma^5 \gamma^\alpha [\gamma_\beta r^\beta + \gamma_\beta q^\beta - m]^{-1} \gamma^\nu [\gamma_\beta r^\beta - \gamma_\beta p^\beta - m]^{-1}$$

$$- ie^2 \int \frac{d^4r}{(2\pi)^4} \operatorname{Tr} \gamma^5 \gamma^\alpha [\gamma_\beta r^\beta + \gamma_\beta q^\beta - m]^{-1} \gamma^\nu [\gamma_\beta r^\beta - m]^{-1} \ . \tag{4.25b}$$

The first and last integrals in (4.25b) vanish because they are two index pseudotensors depending on one vector. The remaining two integrals could be made to cancel against each other if shifts of integration were allowed. Unfortunately such shifts lead to a finite contribution. The value of the surface term evaluated is (see Exercise 4.3)

$$p_\mu T_1{}^{\alpha\mu\nu}(p,q) = \frac{e^2}{4\pi^2} \epsilon^{\alpha\mu\nu\beta} p_\mu q_\beta \ . \tag{4.25c}$$

Therefore

$$p_\mu T_1{}^{\alpha\mu\nu}(p,q\,|\,a) = \frac{e^2}{4\pi^2} \epsilon^{\alpha\mu\nu\beta} p_\mu q_\beta [1 + \frac{a}{2}] \ . \tag{4.26a}$$

Bose symmetry, which has been maintained all along, insures a similar Ward identity in the ν index.

$$q_\nu T_1{}^{\alpha\mu\nu}(p,q\,|\,a) = -\frac{e^2}{4\pi^2} \epsilon^{\alpha\mu\nu\beta} p_\nu q_\beta [1 + \frac{a}{2}] \ . \tag{4.26b}$$

It is seen that the choice for a which insures the vector Ward identity, $a = -2$, is different from the choice that insures the axial Ward identity, $a = 0$. The conclusion is that there is no way of evaluating $T_1^{\alpha\mu\nu}(p,q)$ so that both Ward identities are satisfied. This remarkable result is even more striking when it is remembered that $\Gamma_1^{\alpha\mu\nu}(p,q)$ is not divergent in the explicit evaluation.

One might inquire whether it is possible to add to $T_1^{\alpha\mu\nu}$ a further seagull, which then would restore both Ward identities. If such a seagull were to exist, one would gladly insert it into the definition of $T_1^{\alpha\mu\nu}$ even though it did not arise "naturally" from the integration. It should be clear that no such further additions are possible. Any seagull one adds must be a three-index pseudotensor, and a polynomial in p and q. Bose symmetry limits it to be proportional to $\epsilon^{\alpha\mu\nu\beta}(p_\beta - q_\beta)$. This is precisely the arbitrariness which we have previously allowed for; see (4.22); and it is not sufficient to establish both Ward identities.

Faced with the impossibility of maintaining both Ward identities, we must decide which one we shall accept and which one we shall abandon, i.e., we wish to choose a. It is recognized that the vector Ward identity is a consequence of gauge invariance, while the axial Ward identity is a consequence of an equation of motion, $\partial_\mu J_5^{\ \mu} = F\mu^2 \phi$. Clearly the former is a much more important principle, and a should be set equal to -2. If there were a physical principle which assured the conservation of the axial current as well, we would be faced with a much more problematical situation. Thus we should be grateful that massless neutral pions do not, in fact, occur in nature. We conclude, therefore, that the reason for the violation of the Sutherland-Veltman theorem, $T(0) = 0$, is the violation of the axial Ward identity. Once a modified Ward identity is used, the Sutherland-Veltman theorem is modified, and the new conclusion agrees with the explicit evaluation. With the choice for a which assures gauge invariance, the Ward identities are

$$p_\mu T^{\alpha\mu\nu}(p,q) = q_\nu T^{\alpha\mu\nu}(p,q) = 0 \ , \tag{4.27a}$$

and

$$k_\alpha T^{\alpha\mu\nu}(p,q) = \frac{F\mu^2}{\mu^2 - k^2} T^{\mu\nu}(p,q) - \frac{e^2}{2\pi^2}\epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta \ . \tag{4.27b}$$

The Sutherland-Veltman derivation is now modified at the crucial step (4.6a). Instead of that equation, we have

$$\begin{aligned} \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta T(k^2) &= T^{\mu\nu}(p,q) \\ &= \frac{\mu^2 - k^2}{F\mu^2}\left\{k_\alpha T^{\alpha\mu\nu}(p,q) + \frac{e^2}{2\pi^2}\epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta\right\}. \end{aligned} \tag{4.28}$$

The first term in the brackets is as before; therefore

$$\mathrm{T}(k^2) = \frac{\mu^2 - k^2}{F\mu^2}\left\{k^2\,[F_1(k^2) - F_3(k^2)] + \frac{e^2}{2\pi^2}\right\} ,$$

$$\mathrm{T}(0) = \frac{e^2}{2\pi^2 F} . \tag{4.29}$$

This agrees with the explicit calculations, (4.14).

The phenomenon of the violation of a Ward identity in perturbation theory should be familiar from quantum electrodynamics. For example, the vacuum-polarization tensor and the photon-photon scattering amplitude, as calculated perturbatively in spinor electrodynamics, are not transverse to the photon momenta as they should be. The conventional way of restoring gauge invariance is by the Pauli-Villars regulator technique. It is instructive to demonstrate the workings of that technique in the present context.

Recall that according to the Pauli-Villars regulator method, an amplitude involving a loop integration is considered to be a function of the mass of the particles circulating in the loop. A "regulated" amplitude is defined as the difference between the given amplitude and the same amplitude with the mass evaluated at a "regulator" mass. Finally the physical amplitude is regained by letting the regulator mass pass to infinity. Thus for the pion decay amplitude we have

$$\mathrm{T}^{\mu\nu}_{\mathrm{Reg}}(p,q) = \mathrm{T}^{\mu\nu}(p,q\,|\,m) - \mathrm{T}^{\mu\nu}(p,q\,|\,M) , \tag{4.30a}$$

$$\mathrm{T}^{\mu\nu}_{\mathrm{Physical}}(p,q) = \lim_{M\to\infty} \mathrm{T}^{\mu\nu}_{\mathrm{Reg}}(p,q) . \tag{4.30b}$$

According to (4.14), $\mathrm{T}^{\mu\nu}(p,q\,|\,M)$ vanishes as $\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta(ge^2/4\pi^2 M)$ for large M, hence

$$\mathrm{T}^{\mu\nu}_{\mathrm{Physical}}(p,q) = \mathrm{T}^{\mu\nu}(p,q\,|\,m) . \tag{4.30c}$$

This is as it should be, since $\mathrm{T}^{\mu\nu}(p,q)$ was evaluated unambiguously from a finite integral. For the axial current amplitude on the other hand, we have

$$\mathrm{T}^{\alpha\mu\nu}_{1,\mathrm{Reg}}(p,q\,|\,a) = \mathrm{T}_1{}^{\alpha\mu\nu}(p,q\,|\,a\,|\,m) - \mathrm{T}_1{}^{\alpha\mu\nu}(p,q\,|\,a\,|\,M) , \tag{4.31a}$$

$$\mathrm{T}^{\alpha\mu\nu}_{1,\mathrm{Physical}}(p,q) = \lim_{M\to\infty} \mathrm{T}^{\alpha\mu\nu}_{1,\mathrm{Reg}}(p,q\,|\,a) . \tag{4.31b}$$

Consider now the vector Ward identity. According to (4.26a)

$$p_\mu T^{\alpha\mu\nu}_{1,\mathrm{Reg}}(p,q\,|a) = \frac{e^2}{4\pi^2}\epsilon^{\alpha\mu\nu\beta}p_\mu q_\beta\,[1+\frac{a}{2}] - \frac{e^2}{4\pi^2}\epsilon^{\alpha\mu\nu\beta}p_\mu q_\beta\,[1+\frac{a}{2}] = 0\,, \qquad (4.32a)$$

$$p_\mu T^{\alpha\mu\nu}_{1,\mathrm{Physical}}(p,q) = 0\,. \qquad (4.32b)$$

For the axial Ward identity, we have according to (4.24) and (4.30c)

$$k_\alpha T^{\alpha\mu\nu}_{1,\mathrm{Reg}}(p,q\,|a) = 2mg^{-1}T^{\mu\nu}(p,q\,|m) + \frac{ae^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta - 2Mg^{-1}T^{\mu\nu}(p,q\,|M) - \frac{ae^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta$$

$$= 2mg^{-1}T^{\mu\nu}_{\mathrm{Physical}}(p,q) - 2Mg^{-1}T^{\mu\nu}(p,q\,|M)\,, \qquad (4.33a)$$

$$k_\alpha T^{\alpha\mu\nu}_{1,\mathrm{Physical}}(p,q) = 2mg^{-1}T^{\mu\nu}_{\mathrm{Physical}}(p,q) - \lim_{M\to\infty} 2Mg^{-1}T^{\mu\nu}(p,q\,|M)\,. \qquad (4.33b)$$

Since $\lim_{M\to\infty} 2Mg^{-1}T^{\mu\nu}(p,q\,|M) = \frac{e^2}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta$, we are left with

$$k_\alpha T^{\alpha\mu\nu}_{1,\mathrm{Physical}}(p,q) = 2mg^{-1}T^{\mu\nu}_{\mathrm{Physical}}(p,q) - \frac{e^2}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta\,. \qquad (4.33c)$$

It is seen that the Pauli-Villars technique automatically evaluates the gauge invariant expression for the amplitude. It selects $a = -2$, which, as we have seen, leads to a violation of the axial Ward identity.

In conclusion we remark that the troubles we found with the matrix element of the axial current are not restricted to the σ model. It is clear that it is the triangle graph which leads to difficulties. Such a graph occurs in quantum electrodynamics, in a quark model, indeed in any model which possesses an axial current which is bilinear in fermion fields. This observation will permit us to generalize the present results beyond the specific σ model [6].

4.5 *Anomalous Commutators*

We have seen that the evaluation of the vector, vector, axial vector triangle graph, Fig. 4-2a, results in a formula for $T_1^{\alpha\mu\nu}(p,q)$ which does not satisfy the Ward identities one would naively expect. Our next task is to understand the breakdown of the Ward identities in terms of the anomalous commutators, which must be responsible for this state of affairs [7].

According to the BJL theorem, the ETC between the various currents may be evaluated from the high energy behavior of the triangle graph. Evidently we now must go off the photon mass shell $p^2 = q^2 = 0$, so that the time component of the 4-momentum can be sent to infinity independently, as is required by the BJL technique. It turns out, for our purposes, to be sufficient to go off mass shell for one photon only. Thus we are led to consider

$$\bar{T}_1^{\alpha\mu}(p,q) = -ie\int d^4x\, e^{-ip\cdot x} \langle 0|T J^\mu(x)\bar{J}_5^\alpha(0)|\gamma q\rangle$$

$$= \bar{T}_1^{\alpha\mu\nu}(p,q)\,\epsilon_\nu(q)\ , \tag{4.34a}$$

$$\bar{T}_1^{\alpha\mu\nu}(p,q) = -ie\int d^4x\, d^4y\, e^{-ip\cdot x}\, e^{-iq\cdot y}$$

$$\times\ \langle 0|T J^\mu(x) J^\nu(y)\bar{J}_5^\alpha(0)|0\rangle\Big|_{\substack{q^2=0\\ p^2\neq 0}} \tag{4.34b}$$

Here the bar over $\bar{T}_1^{\alpha\mu}$ and $\bar{T}_1^{\alpha\mu\nu}$ serves to remind us that one photon is on the mass shell. The bar over $\bar{J}_5^\alpha$ indicates that we are not considering the full axial current of the σ model, (4.12a), but only the part bilinear in the nucleon fields; thus the lowest order matrix element involves only the problematical triangle graph, Fig. 4-2a. Note also that we are interested in the T product, *not* the covariant T* product. It is the former object that determines the ETC by the BJL definition.

According to the discussion of Section 3, the following formula for the ETC will enable us to calculate it.

$$\lim_{p_0 \to \infty} p_0 \bar{\mathrm{T}}_1{}^{\alpha\mu}(p,q)$$
$$= -e\int \mathrm{d}^3x\, e^{i\mathbf{p}\cdot\mathbf{x}} \langle 0 | [J^\mu(0,\mathbf{x}), \bar{J}_5{}^\alpha(0)] | \gamma q \rangle \, . \tag{4.35}$$

Our program, therefore, is the following. We evaluate the triangle graph as before, except that the photon with 4-momentum p is not off mass shell. From the explicit formula for that amplitude, which is a covariant T* product, as it must be since it arises from covariant Feynman rules, we extract the non-covariant T product by dropping all seagulls — all polynomials in p_0. This provides us with an explicit formula for $\bar{\mathrm{T}}_1{}^{\alpha\mu}(p,q) = \bar{\mathrm{T}}_1{}^{\alpha\mu\nu}(p,q)\epsilon_\nu(q)$. Note that the present evaluation does not suffer from the ambiguities which beset the calculation of $\mathrm{T}_1{}^{\alpha\mu\nu}(p,q)$ in the previous subsection. The reason is that all the previously encountered ambiguities are seagulls, which we are neglecting.

The evaluation of the relevant triangle graph appears in the literature [8]. The integral yields a finite result as before, and the limit indicated in (4.35) is performed. The resulting commutators are summarized by the following formulas.

$$[J^0(t,\mathbf{x}), J_5^0(t,\mathbf{y})] = 2c\, {}^*F^{0j}(y)\, \partial_j \delta(\mathbf{x}-\mathbf{y}) \, , \tag{4.36a}$$

$$[J^0(t,\mathbf{x}), J_5{}^i(t,\mathbf{y})] = c\, {}^*F^{ij}(y)\, \partial_j \delta(\mathbf{x}-\mathbf{y}) \, , \tag{4.36b}$$

$$[J^i(t,\mathbf{x}), J_5^0(t,\mathbf{y})] = -c\, \partial_j\, {}^*F^{ij}(x)\, \delta(\mathbf{x}-\mathbf{y}) + c\, {}^*F^{ij}(y)\, \partial_j \delta(\mathbf{x}-\mathbf{y}) \, . \tag{4.36c}$$

Here $c = ie/4\pi^2$ and ${}^*F^{\mu\nu}$ is the antisymmetric, conserved electromagnetic dual tensor: ${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)$. In offering (4.36), we do not imply that we have derived the ETC by any operator technique. All that is meant is that the limit (4.35) is non-vanishing and the non-zero expression may be regained by evaluating the appropriate matrix element of (4.36). For example, explicit computation shows that

$$\lim_{p_0 \to \infty} p_0 \bar{\mathrm{T}}_1{}^{00}(p,q) = -\frac{e^2}{2\pi^2}\epsilon^{0\mu\nu\alpha} p_\mu q_\nu \epsilon_\alpha(q) \, . \tag{4.37}$$

The right-hand side of (4.37) is also obtained when (4.36a) is inserted into the right-hand side of (4.35). Therefore, properly speaking, all we have shown is that the ETC (4.36) has non-canonical contributions whose vacuum-one photon matrix element is equal to the same matrix element of the operators appearing in the right-hand side of (4.36).

We see that the ETC between time components has acquired a non-canonical ST, Eq. (4.36a). Therefore according to the discussion of Section 2, the Feynman conjecture may not be satisfied for both Ward identities. To see that indeed the conjecture is violated [9], the ETC (4.36) is expressed in the formalism of Section 2.

$$[J^{\mu}(x), J_5{}^{\nu}(y)]\,\delta([x-y]\cdot n)$$
$$= C^{\mu\nu}(x;n)\,\delta^4(x-y) + S^{\mu\nu|\alpha}(y;n)\,P_{\alpha\beta}\,\partial^{\beta}\delta^4(x-y)\ . \quad (4.38a)$$

The ST, according to (4.36), is

$$S^{\mu\nu|\alpha}(y;n)$$
$$= c\,{}^*F^{\mu\gamma}(y)\,[g_{\gamma}^{\alpha}n^{\nu} + g^{\alpha\nu}n_{\gamma}] + c\,{}^*F^{\nu\gamma}(y)\,[g_{\gamma}^{\alpha}n^{\mu} + g^{\alpha\mu}n_{\gamma}]\ . \quad (4.38b)$$

Equation (4.38b) may be expressed as a total divergence.

$$S^{\mu\nu|\alpha}(y;n)$$
$$= c\,{}^*F^{\mu\gamma}(y)\,\frac{\delta}{\delta n_{\alpha}}[n_{\gamma}n^{\mu}] + c\,{}^*F^{\nu\gamma}(y)\,\frac{\delta}{\delta n_{\alpha}}[n_{\gamma}n^{\mu}]\ . \quad (4.38c)$$

Hence the seagull which covariantizes the T product of J^{μ} and $J_5{}^{\alpha}$ is given by

$$\tau^{\mu\nu}(x,y;n) = \int^{n} S^{\mu\nu|\alpha}(y;n')\,\mathrm{d}n'_{\alpha}\,\delta^4(x-y) + \tau_0{}^{\mu\nu}(x,y)$$
$$= \tau^{\mu\nu}(y;n)\,\delta^4(x-y) + \tau_0{}^{\mu\nu}(x,y)\ ,$$

$$\tau^{\mu\nu}(y;n) = c\,{}^*F^{\mu\gamma}(y)n_{\gamma}n^{\nu} + c\,{}^*F^{\nu\gamma}(y)n_{\gamma}n^{\mu}\ . \quad (4.39)$$

In the above, as before, $\tau_0^{\mu\nu}(x,y)$ is a covariant seagull, as yet undermined. The covariant quantities $I_1^{\mu\nu}$ and $I_2^{\mu\nu}$ defined from $\tau^{\mu\nu}(y;n)$ by (2.47) may now be evaluated.

$$n_\mu \tau^{\mu\nu}(y;n) = n_\mu I_1^{\mu\nu}(y) = c\,{}^*F^{\nu\gamma}(y) n_\gamma \ ,$$

$$n_\nu \tau^{\mu\nu}(y;n) = n_\nu I_2^{\mu\nu}(y) = c\,{}^*F^{\mu\gamma}(y) n_\gamma \ . \tag{4.40a}$$

Evidently we have

$$I_2^{\mu\nu}(y) = -I_1^{\mu\nu}(y) = c\,{}^*F^{\mu\nu}(y) \ . \tag{4.40b}$$

Finally to evaluate the covariant seagull we make use of the relations (2.48). These require that the following combinations be free of gradients of the δ function

$$\frac{\partial}{\partial x^\mu}\tau_0^{\mu\nu}(x,y) - c\,{}^*F^{\mu\nu}(y)\,\partial_\mu\,\delta^4(x-y) \ , \tag{4.41a}$$

$$\frac{\partial}{\partial y^\nu}\tau_0^{\mu\nu}(x,y) - c\,{}^*F^{\mu\nu}(y)\,\partial_\nu\delta^4(x-y) \ . \tag{4.41b}$$

The first of the above equations assures the validity of Feynman's conjecture in the μ, vector Ward identity; while the second effects this state of affairs in the ν, axial Ward identity.

It may be verified that a solution to both conditions (4.41) is

$$\tau_0^{\mu\nu}(x,y)$$
$$= c\,{}^*F^{\mu\nu}(y)\,\delta^4(x-y) + 2c\,\epsilon^{\mu\nu\alpha\beta}A_\alpha(y)\,\partial_\beta\,\delta^4(x-y) \ . \tag{4.42}$$

Hence it appears that Feynman's conjecture can be satisfied in both indices. However, the seagull (4.42) is unacceptable for the following reason. The explicit dependence of the seagull on the vector potential A_α indicates that gauge invariance has been lost. Recall that all the operators, which we are here considering, are to be sandwiched between the vacuum and one-photon state. If one of these operators is the vector potential, then this matrix element will not be gauge invariant. Therefore we must reject this seagull, and content ourselves with one which allows one or the other of the two

Ward identities to satisfy Feynman's conjecture. Such a seagull may easily be shown to be

$$\tau_0{}^{\mu\nu}(x,y) = -(1+a)\, c\, {}^*F^{\mu\nu}(y)\, \delta^4(x-y) \; . \tag{4.43}$$

Here a is an arbitrary parameter, which is determined only when it is decided which Ward identity is to be satisfied. The choice $a = -2$ effects cancellation of Schwinger terms and seagulls in the μ, vector identity; while $a = 0$ performs this service in ν, axial identity. If the former choice is made, one then finds

$$\frac{\partial}{\partial y^\nu} \mathrm{T}^* J^\mu(x) J_5{}^\nu(y) = \mathrm{T}^* J^\mu(x)\, \partial_\nu J_5{}^\nu(y) + 2c\, {}^*F^{\mu\nu}(y)\, \partial_\nu \delta^4(x-y) \; . \tag{4.44}$$

The second term on the right-hand side of (4.44) is the anomaly.

Thus we have understood why the Ward identities are not satisfied; the ETC between J^μ and $J_5{}^\alpha$ departs from its canonical value, acquiring non-canonical contributions. These non-canonical terms are consequences of the intrinsic singularities of local field theory. They have the property that naive current algebraic manipulations become invalid.

4.6 *Anomalous Divergence of Axial Current*

The anomalies of the triangle graph, which we have understood in terms of non-canonical commutators and modified Ward identities, may also be shown to lead to a modified divergence equation of the neutral, gauge invariant axial current [10].

$$\partial_\mu J_5{}^\mu = J_5 + \frac{e^2}{8\pi^2}\, {}^*F^{\mu\nu} F_{\mu\nu} \; . \tag{4.45}$$

Here J_5 is the naive value of the divergence, derived by application of the equations of motion of whatever model we have under consideration.

We consider the fermion part of the axial current.

$$J_5{}^\mu(x) = i\bar{\psi}(x)\, \gamma^5\, \gamma^\mu\, \psi(x) \; . \tag{4.46}$$

Since it is known that the equal time anti-commutator of ψ and $\bar{\psi}$ involves a three-dimensional δ function, we must expect that $\lim_{x\to y} \bar{\psi}(x)\psi(y)$ is singular. Hence the definition (4.46) for $J_5{}^{\mu}(x)$ is necessarily singular. To regulate this singularity, a small separation is introduced in a preliminary definition for $J_5{}^{\mu}$.

$$J_5{}^{\mu}(x|\epsilon) = i\bar{\psi}(x+\epsilon/2)\gamma^5\gamma^{\mu}\psi(x-\epsilon/2) \; . \tag{4.47a}$$

In the presence of electromagnetism, which we shall always consider to be described by an *external* field (i.e., we work to lowest order in electromagnetism), the definition (4.47a) is not gauge invariant. If the electromagnetic potential A^{μ} is replaced by $A^{\mu} + \partial^{\mu}\Lambda$, where Λ is arbitrary, and the fermion fields are allowed to change correspondingly, $\psi(x) \to e^{ie\Lambda(x)}\psi(x)$, then no changes should occur in quantities of physical interest. The formula (4.47a) does not have this property. A modified expression can be constructed which is gauge invariant.

$$J_5{}^{\mu}(x|\epsilon|a) = i\bar{\psi}(x+\epsilon/2)\gamma^5\gamma^{\mu}\psi(x-\epsilon/2)\exp\left(iea\int_{x-\epsilon/2}^{x+\epsilon/2} A^{\alpha}(y)\,\mathrm{d}y_{\alpha}\right) . \tag{4.47b}$$

In (4.47b) a should be set equal to 1 for gauge invariance. However, we prefer to leave this constant unspecified for the time being. The local physical current is obtained by choosing ϵ to be small, averaging over the directions of ϵ and letting $\epsilon^2 = \epsilon_{\mu}\epsilon^{\mu} \to 0$. The method of defining singular products of operators by introducing a small separation is called the "point splitting technique".

We now wish to calculate the divergence of (4.47b). To do so, we need the equation of motion for ψ. We shall here assume that the only interaction is with the external electromagnetic field. More general, interactions have been discussed in the literature [10].

$$i\gamma_{\mu}\partial^{\mu}\psi = m\psi - e\gamma_{\mu}A^{\mu}\psi \; . \tag{4.48}$$

By virtue of (4.48), the divergence of $J_5{}^{\mu}(x|\epsilon|a)$ is

$$\begin{aligned}\partial_{\mu}J_5{}^{\mu}(x|\epsilon|a) &= J_5(x|\epsilon|a) - ieJ_5{}^{\mu}(x|\epsilon|a) \\ &\times\left[A_{\mu}(x+\epsilon/2) - A_{\mu}(x-\epsilon/2) - a\,\partial_{\mu}\int_{x-\epsilon/2}^{x+\epsilon/2} A_{\nu}(y)\,\mathrm{d}y^{\nu}\right] , \\ &= J_5(x|\epsilon|a) - ieJ_5{}^{\mu}(x|\epsilon|a)\epsilon^{\alpha}[\partial_{\alpha}A_{\mu}(x) - a\partial_{\mu}A_{\alpha}(x) + \mathrm{O}(\epsilon)] \; .\end{aligned} \tag{4.49}$$

Here $J_5(x|\epsilon|a)$ is the regulated, split point formula for the naive divergence in this model: $2m\bar{\psi}\gamma^5\psi$. The usual naive result $\partial_\mu J_5{}^\mu = J_5$ is regained from (4.49) if ϵ is set to zero, uncritically. Then the last term in (4.49) appears to vanish. This is legitimate when $J_5{}^\mu(x|\epsilon|a)$ is well behaved as $\epsilon \to 0$. On the other hand, if a matrix element of $J_5{}^\mu(x|\epsilon|a)$ diverges as $\epsilon \to 0$, a finite result may remain. Since the dimension of $J_5{}^\mu$ is $(\text{length})^{-3}$, one may expect a cubic divergence. However, the pseudovector character of $J_5{}^\mu$ reduces the divergence by two powers, leaving a possible linear divergence. We now show that such a divergence is indeed present, and modifies the naive formula for $\partial_\mu J_5{}^\mu(x|\epsilon|a)$.

Consider the vacuum expectation value of $\partial_\mu J_5{}^\mu(x|\epsilon|a)$.

$$\langle 0|\partial_\mu J_5{}^\mu(x|\epsilon|a)|0\rangle = \langle 0|J_5(x|\epsilon|a)|0\rangle$$
$$-ie\,\epsilon^\alpha \langle 0|J_5{}^\mu(x|\epsilon|a)|0\rangle\,[\partial_\alpha A_\mu(x) - a\,\partial_\mu A_\alpha(x) + O(\epsilon)] \quad . \tag{4.50}$$

The vacuum element of $J_5{}^\mu(x|\epsilon|a)$ is non-vanishing because it is computed in the presence of an external electromagnetic field. We have for the last term in (4.50)

$$-i\epsilon^\alpha \langle 0|J_5{}^\mu(x|\epsilon|a)|0\rangle$$
$$= \epsilon^\alpha \langle 0|\bar{\psi}(x+\epsilon/2)\,\gamma^5\gamma^\mu\psi(x-\epsilon/2)|0\rangle \exp\left(iea\int_{x-\epsilon/2}^{x+\epsilon/2} A_\alpha(y)\,\mathrm{d}y^\alpha\right)$$
$$= -\mathrm{Tr}\,\gamma^5\gamma^\mu\epsilon^\alpha \langle 0|\mathrm{T}\psi(x-\epsilon/2)\,\bar{\psi}(x+\epsilon/2)|0\rangle \exp\left(iea\int_{x-\epsilon/2}^{x+\epsilon/2} A^\alpha(y)\,\mathrm{d}y_\alpha\right)$$
$$= -\mathrm{Tr}\,\gamma^5\gamma^\mu\epsilon^\alpha\, G(x-\epsilon/2, x+\epsilon/2) \exp\left(iea\int_{x-\epsilon/2}^{x+\epsilon/2} A^\alpha(y)\,\mathrm{d}y_\alpha\right) \quad . \tag{4.51}$$

The fermion propagator function, $G(x,y)$, in the external field A^μ, has been introduced. In offering (4.51), ϵ^0 is taken to be positive.

$G(x,y)$ possesses an expansion in powers of A^μ which may be summarized graphically as in Fig. 4-3. The double line represents G; the single line is the free fermion propagator $S(x)$; while the $\times$ represents an interaction with the external field. $S(x)$ behaves as $1/x^3$ as $x \to 0$. Therefore the successive terms in the series for $G(x,y)$ behave, when $x \to y$, as $(x-y)^{-3}$, $(x-y)^{-2}$, $(x-y)^{-1}$, $\log(x-y)$, etc. For our calculation of $G(x-\frac{1}{2}\epsilon, x+\frac{1}{2}\epsilon)$ we need terms which do not vanish, for small ϵ, when multiplied by ϵ. Therefore we set

$$\begin{aligned} G(x-\tfrac{1}{2}\epsilon, x+\tfrac{1}{2}\epsilon) &= S(-\epsilon) + ie\int \mathrm{d}^4y\, S(x-\tfrac{1}{2}\epsilon-y)\,\gamma^\alpha\, S(y-x-\tfrac{1}{2}\epsilon) A_\alpha(y) \\ &\quad - e^2\int \mathrm{d}^4y\,\mathrm{d}^4z\, S(x-\tfrac{1}{2}\epsilon-y)\,\gamma^\alpha\, S(y-z)\,\gamma^\beta \\ &\quad \times S(z-x-\tfrac{1}{2}\epsilon) A_\alpha(y) A_\beta(z) + \mathrm{O}(\log\epsilon) \;. \end{aligned} \tag{4.52a}$$

$$S(x) = i\int \frac{\mathrm{d}^4p}{(2\pi)^4}\, e^{-ip\cdot x}\,[\gamma_\alpha p^\alpha - m]^{-1} \;. \tag{4.52b}$$

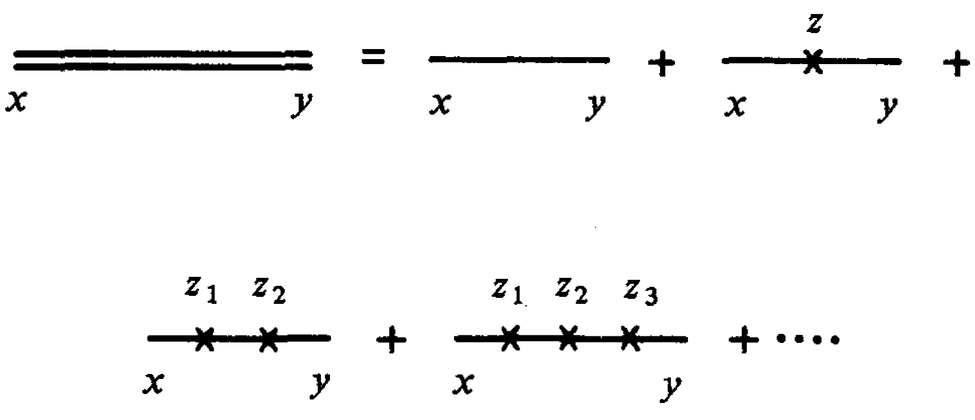

Fig. 4-3. Fermion propagator $G(x,y)$ in an external field.

By C invariance, only the contribution to G which is linear in A^μ is of interest. That term in (4.52a) has the following momentum representation.

$$ie\int \frac{\mathrm{d}^4p\,\mathrm{d}^4q}{(2\pi)^8}\, e^{i\epsilon\cdot p}\, e^{-ix\cdot q}\, S(p+\tfrac{1}{2}q)\,\gamma^\alpha\, S(p-\tfrac{1}{2}q) A_\alpha(q) \;. \tag{4.53}$$

Therefore (4.51) becomes

$$
\begin{aligned}
&-\operatorname{Tr}\left[\epsilon^\alpha \gamma^5 \gamma^\mu G(x-\tfrac{1}{2}\epsilon, x+\tfrac{1}{2}\epsilon)\right] \\
&= -ie \operatorname{Tr} \gamma^5 \gamma^\mu \int \frac{d^4p\, d^4q}{(2\pi)^8} \epsilon^\alpha e^{i\epsilon\cdot p} e^{-ix\cdot q} \\
&\quad \times S(p+\tfrac{1}{2}q)\gamma^\nu S(p-\tfrac{1}{2}q) A_\nu(q) + O(\epsilon \log \epsilon) \\
&= e \operatorname{Tr} \gamma^5 \gamma^\mu \int \frac{d^4p\, d^4q}{(2\pi)^8} e^{i\epsilon\cdot p} e^{-ix\cdot q} A_\nu(q) \frac{\partial}{\partial p_\alpha} \\
&\quad \times S(p+\tfrac{1}{2}q)\gamma^\nu S(p-\tfrac{1}{2}q) + O(\epsilon \log \epsilon) \; .
\end{aligned} \tag{4.54a}
$$

The last equality follows by integration by parts. We now set ϵ to zero. The p-integral is just a surface term; it is easily evaluated by the symmetric methods exemplified in the exercises. The remaining q-integral inverts the Fourier transform of $A_\nu(q)$. The final result for (4.54a) is

$$
-\operatorname{Tr}\left[\epsilon^\alpha \gamma^5 \gamma^\mu G(x-\tfrac{1}{2}\epsilon, x+\tfrac{1}{2}\epsilon)\right]_{\epsilon\to 0} = -\frac{e}{8\pi^2} {}^*F^{\mu\alpha}(x) \, . \tag{4.54b}
$$

Therefore returning to (4.51) and letting ϵ go to zero, we have

$$
\begin{aligned}
&\langle 0 | \partial^\mu J_\mu{}^5(x|a) | 0 \rangle \\
&= \langle 0 | J_5(x|a) | 0 \rangle + \frac{e^2(1+a)}{16\pi^2} {}^*F^{\mu\nu}(x) F_{\mu\nu}(x) \; .
\end{aligned} \tag{4.55}
$$

It is seen that when the gauge invariant definition is selected, $a = 1$, then the divergence of the axial current contains an anomalous term. The naive divergence equation is regained at the expense of gauge invariance if $a = -1$. One may give a simple, heuristic argument which illuminates the origin of this anomalous divergence term. Consider the naive axial current in the model discussed in this subsection. In order to assure gauge invariance, a Pauli-Villars regulator field Ψ is introduced, and correspondingly a regulated axial current is defined.

$$
J_5{}^\mu \Big|_{\text{Reg}} = J_5{}^\mu - \mathcal{J}_5{}^\mu \; . \tag{4.56a}
$$

$\mathcal{J}_5{}^\mu$ is constructed from the regulator fields, in the same fashion as $J_5{}^\mu$ is constructed from the usual fields. The physical axial current is regained by letting the mass M of the regulator field pass to infinity. The divergence of (4.56a) is

$$\partial_\mu J_5{}^\mu \Big|_{\mathrm{Reg}} = 2m\bar{\psi}\gamma^5\psi - 2M\bar{\Psi}\gamma^5\Psi \; . \tag{4.56b}$$

Now when $M \to \infty$, the regulator field contribution to (4.56b) may leave a non-vanishing remainder if the matrix elements of $\bar{\Psi}\gamma^5\Psi$ behave as M^{-1} for large M. Detailed calculation shows that this indeed occurs.

Note that the anomalous divergence does not directly affect our previous derivation of the Sutherland-Veltman theorem for $\pi^0 \to 2\gamma$. The amplitude $\mathrm{T}^{\mu\nu}$, which is considered in (4.3), is already $0(e^2)$, the two photons having been contracted out of the state. Hence to order e^2 we need not inquire into any modification of $\partial_\mu J_5{}^\mu$. The anomaly in that argument came from the commutators and seagull, as was explicitly demonstrated. Nevertheless it is possible to use an anomalous divergence equation to give an alternate derivation of the true Sutherland-Veltman theorem [11]. The photons are not contracted out of their state, and we consider (4.2) in conjunction with (4.45).

$$\begin{aligned} \mathrm{T}^{\mu\nu}(p,q) &= \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta \mathrm{T}(k^2) = (\mu^2 - k^2)\langle 0 | \phi(0) | \gamma, p; \gamma', q\rangle \\ &= \frac{e^2}{8\pi^2 F\mu^2}(k^2-\mu^2)\langle 0 | {}^*F^{\mu\nu}(0)\, F_{\mu\nu}(0) | \gamma, p; \gamma', q\rangle \\ &\quad + \frac{1}{F\mu^2}(\mu^2 - k^2)\,\partial_\alpha \langle 0 | J_5{}^\alpha(0) | \gamma, p; \gamma', q\rangle \; . \end{aligned} \tag{4.57a}$$

The last term in (4.57a) is the divergence of a gauge invariant, three index pseudotensor, hence the original Sutherland-Veltman argument applies. We conclude that it will not contribute to T(0). Note that in this derivation we do not pull the divergence through any T* product, so we need not concern ourselves with commutators. The matrix element of the anomaly may be evaluated to lowest order in electromagnetism. Its value is

$$\begin{aligned} &\frac{e^2}{8\pi^2 F\mu^2}(k^2-\mu^2)\langle 0 | {}^*F^{\mu\nu}(0)\, F_{\mu\nu}(0) | \gamma, p; \gamma', q\rangle \\ &= \frac{e^2}{2\pi^2 F}\,\frac{(\mu^2 - k^2)}{\mu^2}\,\epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta \; . \end{aligned} \tag{4.57b}$$

Therefore T(0), as before, is $e^2/2\pi^2 F$.

This exercise shows that the anomalies in commutators, which are encountered in the original derivation, and the anomalous divergence are two sides of the same coin. One must be present when the other is.

4.7 Discussion

We conclude this treatment of the anomalies of the neutral axial-vector current with a discussion of various disconnected, but important topics.

(1) Consider massless spinor electrodynamics [12]. The present arguments indicate that it is impossible to define a conserved gauge invariant axial current, in spite of the fact that chirality is a symmetry of the theory. Nevertheless there does exist a conserved, gauge invariant axial charge [11]. This charge is constructed as follows. Define

$$\tilde{J}_5^{\,\mu} = J_5^{\,\mu} - \frac{e^2}{4\pi^2} {}^*F^{\mu\nu} A_\nu \ . \tag{4.58a}$$

$J_5^{\,\mu}$ is gauge invariant, but its conservation is broken by the anomaly. On the other hand $\tilde{J}_5^{\,\mu}$ is conserved, but not gauge invariant. The charge Q_5 constructed from $\tilde{J}_5^{\,\mu}$ is time-independent.

$$Q_5 = \int \mathrm{d}^3 x \, \tilde{J}_5^0 (x) \ . \tag{4.58b}$$

Performing a gauge transformation on Q_5 : $\delta A_\nu = \partial_\nu \Lambda$, we see that Q_5 is gauge invariant, even though $\tilde{J}_5^{\,\mu}$ is not.

$$\begin{aligned}
\delta Q_5 &= \int \mathrm{d}^3 x \left(- \frac{e^2}{4\pi^2} {}^*F^{0\nu}(x) \right) \partial_\nu \Lambda(x) \\
&= \int \mathrm{d}^3 x \left(- \frac{e^2}{4\pi^2} {}^*F^{0i}(x) \right) \partial_i \Lambda(x) \\
&= \int \mathrm{d}^3 x \, \frac{e^2}{4\pi^2} \Lambda(x) \, \partial_i {}^*F^{0i}(x) \\
&= 0 \ .
\end{aligned} \tag{4.58c}$$

The conservation and antisymmetry of ${}^*F^{\mu\nu}$ has been used.

Therefore, in spite of the trouble with the *local* axial current, globally axial symmetry can be implemented in the model. This has the consequence that any property of the model, based on axial symmetry, will be maintained in perturbation theory. For example, the anomaly in the divergence cannot be used to generate a mass for the election [13].

In massless electrodynamics, one may describe the anomaly as a clash between two symmetry principles: gauge invariance and chirality. In perturbation theory it is impossible to maintain both, though either one can be satisfied. Such a clash between the conservation of two symmetry currents has been encountered before in the model field theory of spinor electrodynamics in two dimensions [14].

(2) An important question is whether or not higher order effects modify the anomaly. An argument may be given to the end that in spinor electrodynamics and in the σ model, they do not [11]. The argument is as follows: for definiteness we consider the former theory. To fourth order in e, the axial-vector, vector, vector triangle graph has the insertions represented in Fig. 4-4. If the photon integration is carried out *after* the fermion loop integration, then the fermion loop integral is completely convergent. Therefore all shifts of integration, which are required to verify the Ward identities may be performed with impunity. Thus, it is argued, that no anomalies will be present in this or higher orders.

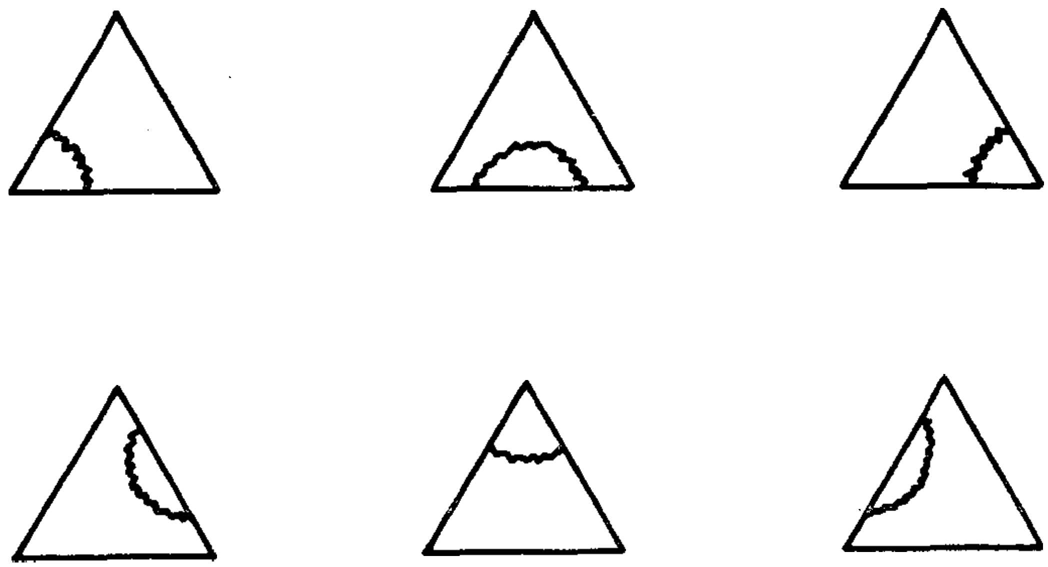

Fig. 4-4. Radiative corrections to the axial-vector, vector, vector triangle graph.

This argument may be criticized because it is somewhat formal [15]. The rules of renormalized perturbation theory require one to perform the photon integrals first, and then the renormalized vertex and propagator corections are to be inserted into the triangle skeleton. In order to resolve this question, explicit calculations for the graphs in Fig. 4-4 have been performed, and they support the formal argument [16]. It should be noticed, however, that the fourth order contribution is characterized by the fact that all indicated corrections are either self-energy or vertex *insertions.* This feature is not true in higher orders, and perhaps something new will be found there [17].

(3) The models in which the axial anomaly has been exposed: spinor electrodynamics and σ model, do not have much dynamical significance for hadron physics. However, as we have stated before, *any* theory with fermion fields out of which the axial vector current is constructed, will possess the anomaly, as long as electromagnetism is coupled minimally. Therefore, it is to be expected that in general PCAC should be modified in the presence of electrodynamics. Thus when we consider the neutral member of the octet of axial currents, $\mathscr{F}_3^{\mu}(x)$, (this current is $\frac{1}{2}$ of the previously defined J_5^{μ}) PCAC should be modified by

$$\partial_\mu \mathscr{F}_3^{\mu} = F\mu^2 \phi_3 + c \frac{e^2}{8\pi^2} {}^*F^{\mu\nu} F_{\mu\nu} . \tag{4.59}$$

Here c is constant which, of course, we cannot derive theoretically in a general fashion. The contribution to c from the triangle graph is determined by the coupling of fermion fields to the axial and vector currents. In the σ model this contribution is $\frac{1}{2}$. In a general quark triplet model where the charges of the quarks are Q, $Q-1$ and $Q-1$, the triangle graph contributes $Q-\frac{1}{2}$ to c [11].

By use of the PCAC hypothesis, and the corrected Sutherland-Veltman theorem, c may be determined experimentally. The currently published $\pi^0 \to 2\gamma$ width of 7.37 ± 1.5 eV sets $|c|$ at 0.44. Further experimental analysis indicates that most likely the sign of c is positive. Thus the theoretical value for c as given by the σ model, triangle graph, $c = \frac{1}{2}$, is in good agreement with the data. If one assumes that in quark models the entire value of c is determined by the triangle graph — a bold hypothesis since one does not know the nature of quark dynamics — $Q = 1$ is preferred, and the conventional quarks with $Q = \frac{1}{3}$ are excluded.

Perhaps at the present stage of understanding of hadron physics, one should not expect to be able to calculate c theoretically. Just as the coefficient of the usual term in $\partial_\mu \mathscr{F}_3^{\mu}$ is taken from experiment — $F\mu^2$ has not been calculated — so also we should content ourselves with an experimental determination of the anomaly.

When c is fitted to the pion data, and a model for SU(3) x SU(3) symmetry breaking is adopted, modified Sutherland-Veltman theorems for $\eta \to 2\gamma$ and

$X \rightarrow 2\gamma$ may be derived. Such calculations have been performed in the context of the $(3, \bar{3}) \oplus (\bar{3}, 3)$ symmetry breaking scheme [18]. The results for the η width are consistent with experiment (~ 1 keV), while the X width comes out remarkably enhanced beyond 80 keV. Present experimental data (< 360 keV) provides no check. Such a check would be very interesting, since this large value is very difficult to understand from any different point of view.

(4) One may wonder why we speak of a modification of PCAC; why one cannot continue using the divergence of the axial current as the pion interpolating field. The answer lies in part in our model calculations where we found that the term $^*F^{\mu\nu}F_{\mu\nu}$ is manifestly not smooth when its matrix elements vary off the pion mass shell. It is this property which we have abstracted, and which we assume holds in nature, as well as in models. Our assumption is supported by the observation that the dimension of the anomaly is 4, and there is no reason to believe that such an operator is smooth. Finally, by adopting the present philosophy, the experimentally unsatisfactory prediction of Sutherland and Veltman is avoided.

(5) The discovery of anomalous Ward identities in the present context engendered a systematic study of all useful Ward identities in SU(3) x SU(3) models [19]. Although other anomalous Ward identities have been found, they seem to be without interest. The only other triangle graph anomaly is in the triple axial vector vertex, which has not, as yet, been used in physical predictions.

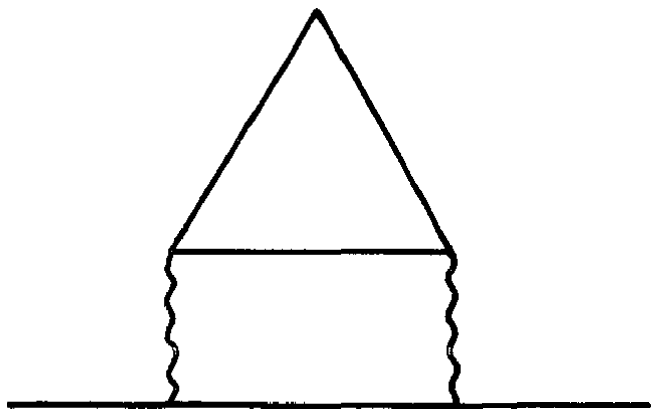

Fig. 4-5. Divergent contribution to axial vector vertex function.

(6) In spinor electrodynamics, the proper electromagnetic vertex function is renormalized by the same infinite constant which effects electron wave function renormalization. This desirable state of affairs is a consequence of the Ward identity satisfied by that function. Before the discovery of anoma-

lous PCAC, it was thought that the proper vertex function of the axial vertex current possesses this property as well, as consequence of the axial Ward identity [20]. The anomaly has destroyed this result; the axial vector vertex function remains infinite after wave function renormalization. The offending graph is the one of Fig. 4-5. One consequence of this is that radiative corrections to neutrino-lepton elastic scattering are infinite [11].

References

[1] D. G. Sutherland, Nucl. Phys. B2 (1967) 433; M. Veltman, Proc. Roy. Soc. A301 (1967) 107.

[2] The argument here follows J. S. Bell and R. Jackiw, Nuovo Cimento 60A (1969) 47.

[3] M. Gell-Mann and M. Levy, Nuovo Cimento 16 (1960) 705.

[4] The evaluation of this graph was first performed by H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. 4 (1949) 347 and J. Steinberger, Phys. Rev. 76 (1949) 1180. Steinberger considered pion decay in the old PS-PS model of π-nucleon interaction. In that theory the pion decays into two photons; the lowest order graphs being given by Fig. 4-1. It presumably is only an accident that this completely implausible calculation gives a result in excellent agreement with experiment.

[5] This careful and unambiguous evaluation is given in Ref. 2.

[6] A historical note is here in order. The first people to calculate the $\pi^0 \to 2\gamma$ process in field theory were H. Fukuda, Y. Miyamoto and J. Steinberger, Ref. 4. In addition to the PS-PS calculation, where the π-N vertex is γ^5, Steinberger also calculated the same amplitude in the PV-PS model, where the π-N vertex is $ik_\alpha\gamma^5\gamma^\alpha$. The second calculation is identical to our Pauli-Villars regulator method evaluation of $T^{\alpha\mu\nu}$. Steinberger then attempted to verify the equivalence theorem between PS-PS and PV-PS theory, which is based on the formal Ward identity (4.16b); and of course failed to do so. He noted this puzzle, and then ceased being a theoretical physicist. Two years later, J. Schwinger, Phys. Rev. 82 (1951) 664, gave an analysis and resolution of the problem. This work was essentially forgotten, and its significance for modern ideas of current algebra and PCAC was not appreciated. The problem was rediscovered in the σ model by J. S. Bell and R. Jackiw, Ref. 2; and independently and simultaneously in spinor electrodynamics by S. L. Adler, Phys. Rev. 177 (1969) 2426. Schwinger's analysis is similar to the one we shall present in subsection 6.

[7] This presentation follows R. Jackiw and K. Johnson, Phys. Rev. 182 (1969) 1459. The same anomalous commutators have also been given by S. L. Adler and D. G. Boulware, Phys. Rev. 184 (1969) 1740.

[8] L. Rosenberg, Phys. Rev. 129 (1963) 2786. Rosenberg's formula appears also in Adler's paper, Ref. 6.

[9] That Feynman's conjecture is violated in $\pi^0 \to 2\gamma$ was first pointed out by R. Jackiw and K. Johnson, Ref. 7. The present analysis follows the paper of D. J. Gross and R. Jackiw, Nucl. Phys. B14 (1969) 269.

[10] Equation (4.45) was first derived by J. Schwinger, Ref. 3. The same form was given by S. L. Adler, Ref. 6, on the basis of an investigation in spinor electrodynamics. The presentation in this section is analogous to Schwinger's and has been given in the contemporary literature by R. Jackiw and K. Johnson, Ref. 7; C. R. Hagen, Phys. Rev. 177 (1969) 2622 and B. Zumino, Proceedings of Topical Conference on Weak Interactions, CERN, Geneva, p. 361 (1969).

[11] This analysis is due to S. L. Adler, Ref. 6.

[12] It is possible that such a theory has physical significance. It has been suggested by K. Johnson, R. Willey and M. Baker, Phys. Rev. 163 (1967) 1699, that the bare mass of the electron is zero and that the physical mass is entirely of dynamical origin.

[13] For example, it might be thought the anomaly allows a mass to be generated in a massless theory without the intervention of Goldstone Bosons. That this is not the case in theories with Abelian vector mesons has been demonstrated by H. Pagels (private communication).

[14] J. Johnson, Phys. Lett. 5 (1963) 253.

[15] R. Jackiw and K. Johnson, Ref. 7.

[16] S. L. Adler and W. Bardeen, Phys. Rev. 182 (1969) 1517.

[17] Ambiguities in Ward identities containing two loop integrations have been found by E. Abers, D. Dicus and V. Teplitz, Phys. Rev. D3 (1971) 485.

[18] S. L. Glashow, R. Jackiw and S. Shei, Phys. Rev. 187 (1969) 1916. An alternate treatment of SU(3) $\times$ SU(3) symmetry breaking together with axial current anomalies has been considered by G. Gounaris, Phys. Rev. D1 (1970) 1426 and Phys. Rev. D2 (1970) 2734. Gounaris' theory makes use of a detailed phenomenological Lagrangian. Correspondingly his results are more precise than those of the above reference, though, of course, they are model dependent. Gounaris gives predictions for off mass shell photon processes, as well as for the processes considered by Glashow, Jackiw and Shei. Agreement is obtained for $\pi^0 \to 2\gamma$ and $\eta \to 2\gamma$; but the prediction for $X^0 \to 2\gamma$ is decreased by a factor of 10.

[19] K. Wilson, Phys. Rev. 181 (1969) 1909; I. Gerstein and R. Jackiw, Phys. Rev. 181 (1969) 1955; W. Bardeen, Phys. Rev. 184 (1969) 1849; R. W. Brown, C. C. Shih and B. L. Young, Phys. Rev. 186 (1969) 1491; D. Amati, G. Bouchiat and J. L. Gervais, Nuovo Cimento 65A (1970) 55.

[20] G. Preparata and W. Weisberger, Phys. Rev. 175 (1968) 1965.

5. Electroproduction Sum Rules

5.1 Preliminaries

In the electroproduction experiments, an electron is scattered off a nucleon target, typically a proton. The hadronic final states that are observed, are thought to arise, in the context of lowest order electromagnetism, from the inelastic interaction between an off mass shell photon and the proton. The process is depicted in Fig. 5-1. By measuring total inelastic cross sections, that is by summing over all final states, one obtains a determination of the