my presentation. I urge you to do them! Some solutions to the Exercises can be found in a separate section.

2. Description of Vector Gauge Theories

2.1 Action and Equations of Motion

We shall be dealing with field theories whose basic dynamical variables are vector fields $A^a_\mu(x)$, called potentials. The index μ refers to Minkowski space-time where the coordinate vector $x^\mu \equiv (t, \mathbf{r})$ is defined with metric $g_{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1)$; we shall deal not only with the physical, four-dimensional world, but also with models in two and three dimensions — these are useful for physical and pedagogical reasons. Throughout, the velocity of light is set to unity; however $\hbar$ will be retained so that classical effects may be clearly separated from quantum ones. The index "a" on the potential labels internal degrees of freedom, and the theory is invariant against a (compact and in general non-Abelian) group of transformations operating on these degrees of freedom. The index a ranges over the dimension of the group: for SO(3) or SU(2) it goes from 1 to 3; for SU(3), it ranges to 8; while for the Abelian case a is single valued, and is suppressed. (For us, upper and lower internal symmetry indices are equivalent.)

From the potential (also called connection by the mathematically minded) the field strength $F^a_{\mu\nu}$ (curvature) is constructed by the formula

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \quad . \tag{2.1}$$

Here ∂_μ is the derivative with respect to x^μ, f_{abc} are completely anti-symmetric structure constants of the group, g is a coupling constant, and repeated indices are summed.

There exists a useful matrix notation for the gauge theory. Consider anti-hermitian representation matrices T^a for the Lie algebra of the invariance group,

$$[T^a, T^b] = f_{abc} T^c, \qquad (T^a)^\dagger = -T^a \quad , \tag{2.2}$$

normalized, for example, by

$$\mathrm{tr}\, T^a T^b = -\tfrac{1}{2}\delta_{ab} \quad , \tag{2.3}$$

and define a matrix-valued vector potential by

$$A_\mu \equiv g T^a A_\mu^a \quad , \tag{2.4a}$$

from which the components can be regained with the help of Eq. (2.3):

$$A_\mu^a = -\frac{2}{g} \operatorname{tr} T^a A_\mu \quad . \tag{2.4b}$$

[For SU(2), $T^a = \sigma^a/2i$, where the σ^a are Pauli matrices; for SU(3) the 3 x 3 Gell-Mann matrices $\lambda^a/2i$ are used.] The matrix-valued field strength

$$F_{\mu\nu} \equiv g T^a F_{\mu\nu}^a \quad , \tag{2.5a}$$

$$F_{\mu\nu}^a = -\frac{2}{g} \operatorname{tr} T^a F_{\mu\nu} \quad , \tag{2.5b}$$

is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad . \tag{2.6}$$

This compact notation encompasses the Abelian (Maxwell) theory (electrodynamics), where the matrices reduce to numbers, and the commutators vanish. When the matrix structure is present, the commutators are nonvanishing and the theory is called non-Abelian. [We emphasize that the commutators in Eq. (2.6) are considered only in the matrix space of the group generators; they are not quantum mechanical.]

Exercise 2.1. Convince yourself that any representation matrices, not only those of the defining representation, may be used for the matrix notation. Of course if their normalization differs from (2.3), the inversion formulas (2.4b) and (2.5b) are correspondingly modified.

The matrix and component notations will be used interchangeably; no confusion arises because the context leaves the convention unambiguous. For the Abelian theory, where the two collapse into one, I shall remain with the real "component" notation for potentials and fields.

We also introduce the gauge covariant derivative D_μ: operating on a matrix quantity $M = M^a T^a$ it gives

$$D_\mu M = \partial_\mu M + [A_\mu, M] \quad . \tag{2.7a}$$

In the gauge covariant derivative, the ordinary derivative is supplemented by a commutator, which however vanishes in the Abelian theory so that covariant and ordinary differentiation coincide. Equation (2.7a) may also be presented in component notation:

$$(D_\mu M)^a = \partial_\mu M^a + g f_{abc} A_\mu^b M^c \quad . \tag{2.7b}$$

The covariant derivative is distributive,

$$D_\mu (MM') = (D_\mu M)M' + M(D_\mu M') \quad , \tag{2.8a}$$

and when operating on functions rather than matrices it reduces to the ordinary derivative; for example

$$\partial_\mu (\operatorname{tr} MM') = \operatorname{tr} (D_\mu M)M' + \operatorname{tr} M(D_\mu M') \quad . \tag{2.8b}$$

Exercise 2.2. Show that

$$(D_\mu D_\nu - D_\nu D_\mu) M = [F_{\mu\nu}, M]. \tag{E2.1}$$

Exercise 2.3. Suppose A_μ is changed by an infinitesimal amount, $A_\mu \to A_\mu + \delta A_\mu$. Show that the first-order change in $F_{\mu\nu}$ is

$$\delta F_{\mu\nu} = D_\mu \delta A_\nu - D_\nu \delta A_\mu . \tag{E2.2}$$

Let us observe that from its definition (2.1) or (2.6) $F_{\mu\nu}$ satisfies an identity, called the Bianchi identity:

$$D_\alpha F_{\beta\gamma} + D_\beta F_{\gamma\alpha} + D_\gamma F_{\alpha\beta} = 0 \quad . \tag{2.9}$$

This is anti-symmetric in its three indices α, β, γ; hence in two dimensions it is vacuous. In higher dimensions, the formula may be presented more compactly by defining the dual field strength with the help of the totally anti-symmetric tensor appropriate to the dimensionality of space-time. In four dimensions, the dual is again a second-rank tensor,

$$*F^{\mu\nu} = \tfrac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \quad , \tag{2.10a}$$

$$F_{\alpha\beta} = -\tfrac{1}{2} \epsilon_{\alpha\beta\mu\nu} {*F}^{\mu\nu} \quad , \quad \epsilon^{0123} = 1 = -\epsilon_{0123} \quad , \tag{2.10b}$$

and the Bianchi identity (2.9) becomes

$$D_\mu {}^*F^{\mu\nu} = 0 \quad . \tag{2.11}$$

In three dimensions the dual is a vector,

$$^*F^\mu = \tfrac{1}{2}\epsilon^{\mu\alpha\beta} F_{\alpha\beta} \quad , \tag{2.12a}$$

$$F_{\alpha\beta} = \epsilon_{\alpha\beta\mu} {}^*F^\mu \quad , \qquad \epsilon^{012} = 1 = \epsilon_{012} \quad , \tag{2.12b}$$

and the Bianchi identity (2.9) requires the dual field to be covariantly conserved,

$$D_\mu {}^*F^\mu = 0 \quad . \tag{2.13}$$

Finally in two dimensions, the dual is a scalar, which is unconstrained since (2.9) becomes vacuous.

$$^*F = \tfrac{1}{2}\epsilon^{\mu\nu} F_{\mu\nu} \quad , \tag{2.14a}$$

$$F_{\mu\nu} = -\epsilon_{\mu\nu} {}^*F \quad , \qquad \epsilon^{01} = 1 = -\epsilon_{01} \quad . \tag{2.14b}$$

We have still to provide dynamical field equations which govern the space-time behavior of our variables. [The above Eqs. (2.9) — (2.13) are not dynamical; they are identities.] The equations of motion may be presented in terms of an action I, which is a functional of A_μ. The requirement that I be stationary against variations of A_μ gives Euler-Lagrange equations which are the field equations for A_μ.

The Yang-Mills theory results by taking the action to be the space-time integral of a Lagrangian (density) $\mathscr{L}_{\mathrm{YM}}$, the latter being the simplest local invariant function constructed from $F_{\mu\nu}$ [4]:

$$I_{\mathrm{YM}} = \int \mathrm{d}x \, \mathscr{L}_{\mathrm{YM}} \quad ,$$

$$\mathscr{L}_{\mathrm{YM}} = \frac{1}{2g^2} \operatorname{tr} F^{\mu\nu} F_{\mu\nu} = -\frac{1}{4} F^{\mu\nu a} F^a_{\mu\nu} \quad . \tag{2.15}$$

The Yang-Mills field equations then read

$$\frac{\delta I_{\mathrm{YM}}}{\delta A^a_\mu} = 0 \Leftrightarrow D_\mu F^{\mu\nu} = 0 \quad , \tag{2.16a}$$

or in component from

$$\partial_\mu F^{\mu\nu a} + g f_{abc} A^b_\mu F^{\mu\nu c} = 0 \quad . \tag{2.16b}$$

We see that even in the absence of other matter couplings, the non-Abelian theory is non-trivial, nonlinear and interacting. (Further possible contributions to the gauge field action, even in the absence of matter fields, as well as interaction with matter fields, will be discussed later.)

It is clear that the non-Abelian Yang-Mills theory is a generalization, into non-commuting matrix-valued potentials and fields, of the Abelian Maxwell theory, which in Yang-Mills terminology we call an Abelian U(1) or SO(2) gauge theory. Notice however an important difference. The Maxwell theory's dynamics can be entirely formulated in terms of field strengths: the Maxwell equations are (the Abelian analogs of) (2.9) or (2.11) and (2.16), while the electromagnetic vector potentials are conventionally introduced to solve those Maxwell equations which coincide with the Bianchi identities, (2.9) or (2.11). Thus there is a historic prejudice that potentials are unphysical, secondary quantities and only field strengths are physically important. However, this attitude is unwarranted in view of modern developments. Already for electromagnetism, one knows that vector potentials are physically significant within quantum mechanics (Bohm-Aharanov effect), while the Yang-Mills theory cannot even be formulated without reference to the potentials, since the covariant derivatives, occurring in field equations, involved A_μ. Other similarities and differences between Maxwell and Yang-Mills theories will be mentioned later.

2.2 Symmetries

Next let us discuss invariance symmetries of the action (2.15); i.e. we consider transformations which take one solution of the field equations (2.16) into another.

The most important symmetry of the Yang-Mills gauge theory is its gauge invariance; indeed the particular form taken for the dynamics is chosen so that symmetry is respected. Let U be a position-dependent element of the group which transforms the internal symmetry degrees of freedom of the theory, and let U be given in the same representation as the matrices T^a. [For SU(2), U is a 2×2 unitary matrix with unit determinant; similarly for SU(3), where U is 3×3.] Then a gauge transformation of the vector potential matrix A_μ is defined as

$$A_\mu \to A_\mu^U \equiv U^{-1} A_\mu U + U^{-1} \partial_\mu U = A_\mu + U^{-1} D_\mu U \ , \quad U^\dagger = U^{-1} \ . \tag{2.17a}$$

This induces the following similarity transformation on the field strengths:

$$F_{\mu\nu} \to F_{\mu\nu}^U = U^{-1} F_{\mu\nu} U \quad . \tag{2.17b}$$

One easily verifies that this is a symmetry transformation of the field Eq. (2.16), i.e., if A_μ, $F_{\mu\nu}$ solve them, so do A_μ^U, $F_{\mu\nu}^U$ for arbitrary U. Correspondingly the action (2.15) is invariant, as also is the Lagrangian since $\mathscr{L}_{\text{YM}}$ involves a trace which is invariant against similarity transformations.

In the above, we have discussed finite gauge transformations. One may also consider infinitesimal ones. If U is written as

$$U = \mathrm{e}^{\Theta} \quad , \tag{2.18a}$$

and expanded in powers of Θ, which is anti-Hermitian,

$$U = I + \Theta + \ldots, \quad \Theta^\dagger = -\Theta \quad , \tag{2.18b}$$

then the gauge transformation to first order in Θ reads

$$A_\mu \to A_\mu + \delta A_\mu \quad , \quad \delta A_\mu = D_\mu \Theta \quad , \tag{2.19a}$$

$$F_{\mu\nu} \to F_{\mu\nu} + \delta F_{\mu\nu} \quad , \quad \delta F_{\mu\nu} = [F_{\mu\nu}, \Theta] \quad , \tag{2.19b}$$

[Compare this with Eqs. (E2.1) and (E2.2).] Equation (2.19) becomes in component notation, $\Theta = \theta^a T^a$

$$\delta_\theta A_\mu^a = \frac{1}{g} \partial_\mu \theta^a + f_{abc} A_\mu^b \theta^c \quad , \tag{2.20a}$$

$$\delta_\theta F_{\mu\nu}^a = f_{abc} F_{\mu\nu}^b \theta^c \quad . \tag{2.20b}$$

These transformations are called "local" gauge transformations, because U and Θ are taken to be local matrix functions of the space-time coordinates.

When they are constant, the transformation reduces to a conventional internal symmetry transformation, also frequently described as a "global" gauge transformation. This is something of a misnomer; I prefer the name "rigid" gauge transformation.

One immediate and important consequence of local gauge invariance is the absence from the Lagrangian of a mass term for A_μ: $M^2 \operatorname{tr} A_\mu A^\mu$ is not locally gauge invariant.

The conserved Noether current j^μ for an invariant Lagrangian $\mathcal{L}$, which satisfies the invariance condition that must be established without use of equations of motion

$$\delta \mathcal{L} \equiv \frac{\partial \mathcal{L}}{\partial A_\nu^a} \, \delta A_\nu^a + \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu^a} \, \partial_\mu \delta A_\nu^a = 0 \quad , \tag{2.21a}$$

is given by

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu^a} \, \delta A_\nu^a \quad . \tag{2.21b}$$

For local gauge transformations, (2.19) and (2.20), the conserved current is

$$j_\theta^\mu = \frac{2}{g^2} \operatorname{tr} F^{\mu\nu} D_\nu \Theta \quad , \tag{2.22}$$

while for rigid transformations, it reads

$$j_a^\mu = -f_{abc} F^{\mu\nu b} A_\nu^c \quad , \tag{2.23}$$

where I am using the component notation and have cancelled away the constant transformation parameter θ^a.

Exercise 2.4. With the help of the field equations (2.16) show that both j_θ^μ in (2.22) and j_a^μ in (2.23) are conserved. Hint: use (2.8b) and (E2.1). The conservation of these currents is of course expected, since the Lagrangian (2.15) is invariant against these transformations: $\delta_\theta \mathcal{L} = 0$.

Obviously non-Abelian gauge invariance is a generalization of familiar electromagnetic gauge invariance, where $U = \exp(i\theta/e)$ and both U and θ are functions, not matrices, thus reducing (2.17) and (2.19) to

$$A_\mu \to A_\mu + \frac{1}{e} \partial_\mu \theta \quad , \tag{2.24a}$$

$$F_{\mu\nu} \to F_{\mu\nu} \quad . \tag{2.24b}$$

(An electromagnetic coupling constant e is conventionally inserted, even though the free Maxwell theory is without interaction.)

However there are important differences between Abelian and non-Abelian gauge transformations: The electromagnetic gauge transformation (2.24) is the same both in its infinite and infinitesimal form, but for the Yang-Mills theory the two differ; compare (2.17) and (2.19). While iterating infinitesimal transformations produces finite transformations, not all finite non-Abelian transformations can be reached in this way. An equivalent statement is that there are finite gauge transformations U which cannot be continuously deformed to the identity I. This fact has far-reaching consequences for the topological structure of non-Abelian gauge theories, and I shall elaborate on this extensively in subsequent lectures.

Another important difference between the Abelian and non-Abelian theories is that the Yang-Mills field strength is not gauge invariant. As is seen from (2.17b), $F_{\mu\nu}$ transforms by a similarity transformation; or equivalently from (2.20b), $F^a_{\mu\nu}$ transforms according to the adjoint representation. We say that $F_{\mu\nu}$ is gauge covariant. This highlights once again the fact that field strengths are not fundamental to the theory. Indeed one cannot in general determine uniquely the potential, even up to gauge transformations, which gives rise to a specific field strength; i.e. gauge non-equivalent A_μ's can lead to the same $F_{\mu\nu}$ [5]. In spite of the more complicated way that gauge transformations operate in a non-Abelian gauge theory, we will insist that all physical quantities be gauge invariant, just as in the Abelian theory.

Exercise 2.5 Consider the charge constructed from the time components of the conserved current (2.23):

$$Q^a = \int \mathrm{d}r\, j^0_a(t, r) \quad . \tag{E2.3}$$

Show that Q^a is time-dependent, provided j_a falls off sufficiently rapidly at large r. Since j_a is not gauge invariant, the fall-off requirement restricts the large r behavior of gauge transformations. While the current j^μ_a has no simple gauge transformation properties, show that Q^a is gauge covariant against gauge transformations U which approach a definite angle-independent limit as r approaches infinity. Hint: use the time component of the Yang-Mills equation to express Q^a as an integral over a surface at spatial infinity.

Exercise 2.6. Show that no new charges, beyond Q^a, arise by integrating over space the time component of j^μ_θ, (2.22), provided Θ approaches an angle-independent limit at spatial infinity.

Gauge invariance of the theory may be used to set one space-time component of the vector potential to zero by appropriate choice of U and without loss of physical content. While this will be useful in our discussion of the quantized theory's Hamiltonian structure, it has a very dramatic effect on two-dimensional Yang-Mills theory: when one component of the vector potential is zero, the commutator $[A^\mu, A^\nu]$ vanishes and the theory becomes linear and trivial, just like two-dimensional, free electromagnetism.

Thus far I have discussed symmetries of Yang-Mills theory associated with transformations of internal degrees of freedom; of course there are also symmetries associated with transformations of space-time coordinates x^μ. The infinitesimal action of these transformations on coordinates

$$x^\mu \to x^\mu + \delta_f x^\mu \quad , \qquad \delta_f x^\mu = -f^\mu(x) \quad , \tag{2.25}$$

induces a transformation on potentials, which conventionally is given by a Lie derivative:

$$\delta_f A_\mu = f^\alpha \partial_\alpha A_\mu + (\partial_\mu f^\alpha) A_\alpha \equiv L_f A_\mu \quad . \tag{2.26}$$

A Lie derivative of a tensor $T^{\nu\cdots}_{\mu\cdots}$ with upper and lower indices is defined by

$$\begin{aligned} L_f T^{\nu\cdots}_{\mu\cdots} \equiv f^\alpha \partial_\alpha T^{\nu\cdots}_{\mu\cdots} \\ + (\partial_\mu f^\alpha) T^{\nu\cdots}_{\alpha\cdots} + \ldots \\ -(\partial_\alpha f^\nu) T^{\alpha\cdots}_{\mu\cdots} - \ldots \quad , \end{aligned} \tag{2.27}$$

where the omitted terms are a repetition of the first, for each index. Note that in general $L_f A_\mu \neq g_{\mu\nu} L_f A^\nu$.

Exercise 2.7. Show that the transformation of the field strength which follows from that for the potential [cf. (E2.2)],

$$\delta_f F_{\mu\nu} = D_\mu \delta_f A_\nu - D_\nu \delta_f A_\mu \quad , \tag{E2.4}$$

is also given by the Lie derivative,

$$\delta_f F_{\mu\nu} = L_f F_{\mu\nu} \quad . \tag{E2.5}$$

For the Minkowski spaces on which our theories are defined, the coordinate transformations of space-time translation and Lorentz rotation are symmetry operations of the Yang-Mills theory. Together they form the Poincaré group, and their infinitesimal action is give by

$$\text{Translation } f^\mu = a^\mu \quad , \tag{2.28}$$

$$\text{Lorentz rotation } f^\mu = \omega^\mu{}_\nu x^\nu \quad , \qquad \omega^\mu{}_\nu = -\omega_\nu{}^\mu \, . \tag{2.29}$$

Here a^μ and $\omega^\mu{}_\nu$ are constant infinitesimal parameters of the transformation. Note that for these $L_f A_\mu = g_{\mu\nu} L_f A^\nu$ and the f^μ's are Killing vectors, i.e., they satisfy the Killing equation,

$$\partial_\mu f_\nu + \partial_\nu f_\mu = 0 \quad . \tag{2.30}$$

Exercise 2.8 The Lie bracket of two vector functions f^μ and g^μ is defined by

$$f^\alpha \partial_\alpha g^\mu - g^\alpha \partial_\alpha f^\mu \equiv h^\mu \quad . \tag{E2.6}$$

Show that the commutator of two coordinate transformations (2.25) satisfies

$$[\delta_f, \delta_g] = -\delta_h \quad , \tag{E2.7}$$

and the same is true for the transformations of the potentials (2.26). [N.B.: $\delta_f \partial_\alpha A_\mu \equiv \partial_\alpha \delta_f A_\mu \neq L_f(\partial_\alpha A_\mu)$.] Show that the Lie bracket of two Killing vectors is again a Killing vector. With (2.28) and (2.29) verify that Lie bracketing these transformations reproduces the commutators of the Poincaré group.

Under the Poincaré transformations (2.28) and (2.29), the Lagrangian changes by a total divergence,

$$\delta_f \, \mathscr{L}_{\text{YM}} = \partial_\mu \Omega_f^\mu \quad , \tag{2.31a}$$

$$\Omega_f^\mu = f^\mu \mathscr{L}_{\text{YM}} \quad . \tag{2.31b}$$

The action changes only by surface terms so that the equations of motion are invariant; i.e., the Yang-Mills theory is Poincaré invariant.

The derivation of the conserved Poincaré currents is somewhat awkward. If one applies directly Noether's theorem to a Lagrangian $\mathscr{L}$ which is not invariant, but changes by a total derivative $\partial_\mu \Omega^\mu$,

$$\delta \mathscr{L} = \partial_\mu \Omega^\mu \quad , \tag{2.32a}$$

then the formula for the conserved current generalizes from (2.21) to [6]

$$j^{\mu} = \frac{\partial \mathscr{L}}{\partial \partial_{\mu} A_{\nu}^{a}} \delta A_{\nu}^{a} - \Omega^{\mu} \quad . \tag{2.32b}$$

[Just as in (2.21a), Eq. (2.32a) must be established without using equations of motion.] In the present case this produces a gauge non-invariant result, since δA_{ν}^{a} is not gauge covariant, and the Noether currents are not expressed in terms of the conserved, symmetric and gauge invariant energy-momentum tensor.

$$\begin{aligned}\theta^{\mu\nu} &= -F^{\mu\alpha a} F^{\nu a}_{\alpha} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta a} F^{a}_{\alpha\beta} \\ &= \frac{2}{g^{2}} \operatorname{tr} (F^{\mu\alpha} F^{\nu}_{\alpha} - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}) \quad .\end{aligned} \tag{2.33}$$

Exercise 2.9. By using the equations of motion (2.16) prove that $\partial_{\mu}\theta^{\mu\nu} = 0$. Hint: Equation (2.8b) and the Bianchi identity (2.9) will simplify the calculation.

However, there exists a well known procedure for "improving" the Noether currents by adding to them superpotentials — divergences of anti-symmetric tensors $\partial_{\nu} X^{\mu\nu}$, $X^{\mu\nu} = -X^{\nu\mu}$ — which do not spoil current conservation, since $\partial_{\mu}\partial_{\nu} X^{\mu\nu} = 0$ [6]. (Ω^{μ} is not uniquely defined by the equation $\partial_{\mu}\Omega^{\mu} = \delta\mathscr{L}$!) With such improvement, the conserved Poincaré currents are

$$j_{f}^{\mu} = \theta^{\mu\nu} f_{\nu} \quad , \tag{2.34}$$

with f_{ν} given by (2.28) or (2.29). This is the so-called Bessel-Hagen form for the current [7].

Exercise 2.10. Find the superpotentials that must be added to the Noether current to arrive at (2.34). The possibility of finding these superpotentials is not accidental, but arises from the Lorentz invariance of the theory. For a general discussion see ref. [6].

By using some geometrical properties of gauge fields, one can arrive at (2.34) by an alternate, more elegant, method, which is particular to a gauge theory. Let us first observe that one can write $\delta_{f} A_{\mu} \equiv L_{f} A_{\mu}$ in terms of a gauge covariant quantity and an (infinitesimal) gauge transformation.

$$L_{f} A_{\mu} = f^{\alpha}\partial_{\alpha} A_{\mu} + (\partial_{\mu} f^{\alpha}) A_{\alpha} = f^{\alpha} F_{\alpha\mu} + D_{\mu}(f^{\alpha} A_{\alpha}) \quad . \tag{2.35}$$

Since the Yang-Mills theory is invariant against gauge transformations, an (infinitesimal) coordinate transformation supplemented by an (infinitesimal) gauge transformation, with gauge function $-f^\alpha A_\alpha$, is still a symmetry transformation. Therefore, we can define the action of coordinate transformations on gauge potentials by an alternate rule, which is gauge covariant [8].

$$\bar{\delta}_f A_\mu \equiv L_f A_\mu - D_\mu(f^\alpha A_\alpha) = f^\alpha F_{\alpha\mu} \quad . \tag{2.36}$$

When Noether's theorem is used in conjunction with this gauge covariant transformation, the Bessel-Hagen current (2.34) is obtained immediately.

Exercise 2.11. Show that

$$\bar{\delta}_f F_{\mu\nu} \equiv D_\mu \bar{\delta}_f A_\nu - D_\nu \bar{\delta}_f A_\mu = f^\alpha D_\alpha F_{\mu\nu} + \partial_\mu f^\alpha F_{\alpha\nu} + \partial_\nu f^\alpha F_{\mu\alpha} \quad . \tag{E2.8}$$

This may be called the gauge covariant Lie derivative. Hint: use the Bianchi identity.

Exercise 2.12. Show that the gauge covariant transformations on the potentials do not follow (E2.7), rather one has

$$[\bar{\delta}_f, \bar{\delta}_g] A_\mu = -\bar{\delta}_h A_\mu - D_\mu(f^\alpha g^\beta F_{\alpha\beta}) \quad . \tag{E2.9}$$

For the finite version of these transformations see ref. [8].

In our discussion of symmetries no attention has been paid to the fact that one is dealing with quantum field operators, rather than classical c-number fields. There are two sources of differences between the two, which could invalidate the results here presented. First, there is the problem of operator ordering — in all our manipulations we ignored the quantum mechanical non-commutativity of the quantities with which we were working. Second, a difficulty specific to quantum field theories is the occurrence of infinities and hence the necessity of regularizing and renormalizing the theory. This in general produces further terms in the action, equations of motion etc. — the counter terms — which could spoil the symmetries that we have discussed. (When speaking of symmetries in the quantum field theory, we consider the parameters of the transformation — e.g. θ^a for gauge transformations, a^μ, ω^μ_ν for Poincaré transformations — to be c-numbers.)

An analysis of these questions requires a course in renormalization theory, well beyond the scope of my lectures. Let it suffice to state that the Yang-Mills model presented here can be quantized so that gauge invariance and Poincaré invariance are indeed preserved. However, as soon as additional couplings to other fields are included, it may not be possible to maintain the symmetries, and we shall discuss this below.

Whenever a classical theory possesses a symmetry, but there is no way of quantizing the theory to preserve that symmetry, we say that there are "anomalies" in the conservation equations for the symmetry currents. I am telling you, without proof, that for Yang-Mills theory there are no known Poincaré or gauge anomalies.

An example of anomalies is found in four-dimensional Yang-Mills theory, which on the classical level possesses a larger space-time symmetry than Poincaré invariance. Observe that the divergence of the Bessel-Hagen current (2.34) with arbitrary f^{μ} can be written as

$$\partial_{\mu} j_f^{\mu} = (\partial_{\mu} \theta^{\mu\nu}) f_{\nu} + \theta^{\mu\nu} \partial_{\mu} f_{\nu} = \tfrac{1}{2} \theta^{\mu\nu} (\partial_{\mu} f_{\nu} + \partial_{\nu} f_{\mu}) \quad . \tag{2.37}$$

In passing from the first equality to the second we have used the symmetry of $\theta^{\mu\nu}$ in (μ, ν) and the conservation of $\theta^{\mu\nu}$; see Exercise 2.9. For a Killing vector (2.30), in any number of dimensions, the term in parentheses vanishes; j_f^{μ} is conserved and this is the previously discussed Poincaré invariance. Maintaining this symmetry on the quantum level is equivalent to constructing a conserved, symmetric, renormalized energy-momentum tensor and this can indeed be done [9].

In four dimensions the formal (i.e. classical, not quantized and renormalized) energy-momentum tensor (2.33) is also trace-free in the (μ, ν) indices. Hence (2.37) may also be rewritten as

$$\partial_{\mu} j_f^{\mu} = \tfrac{1}{2} \theta^{\mu\nu} (\partial_{\mu} f_{\nu} + \partial_{\nu} f_{\mu} - \tfrac{1}{2} g_{\mu\nu} \partial_{\alpha} f^{\alpha}) \quad . \tag{2.38}$$

Consequently if the infinitesimal coordinate transformation f^{μ} satisfies the four-dimensional conformal Killing equation,

$$\partial_{\mu} f_{\nu} + \partial_{\nu} f_{\mu} - \tfrac{1}{2} g_{\mu\nu} \partial_{\alpha} f^{\alpha} = 0 \quad , \tag{2.39}$$

j_f^{μ} will be conserved, as a consequence of the theory's invariance against these further coordinate transformations. Vector functions solving (2.39) are called conformal Killing vectors; every Killing vector is a conformal Killing one, but one can find other conformal Killing vectors. The solutions of (2.39) beyond (2.28) and (2.29) correspond to dilatations and special conformal transformations:

$$\text{Dilatation} \quad f^{\mu} = a x^{\mu} \quad , \tag{2.40}$$

$$\text{Special conformal transformation } f^\mu = 2c \cdot x x^\mu - c^\mu x^2 \quad . \tag{2.41}$$

Here a and c^μ are infinitesimal parameters.

Exercise 2.13. Show that (2.28), (2.29), (2.40) and (2.41) are the only solutions to (2.39) in four dimensions. Is the same true in three dimensions, in two dimensions? The conformal Killing equation in d dimensions reads

$$\partial_\mu f_\nu + \partial_\nu f_\mu - \frac{2}{d} g_{\mu\nu} \partial_\alpha f^\alpha = 0 \quad . \tag{E2.10}$$

The finite version of these transformation is

$$\text{Dilatation } x^\mu \to x'^\mu = e^{-a} x^\mu \quad . \tag{2.42}$$

$$\text{Special conformal transformation } x^\mu \to x'^\mu = \frac{x^\mu + c^\mu x^2}{1 + 2c \cdot x + c^2 x^2} \tag{2.43}$$

One readily verifies that if the A_μ's are transformed by the rule (2.26) or (2.36), the Lagrangian changes as in (2.31) and Noether's theorem, combined with (2.36), again yields the Bessel-Hagen conserved current (2.34).

Exercise 2.14. Show that $\bar{\delta}_f \mathscr{L}_{YM} = \partial_\mu \Omega^\mu_f$, $\Omega^\mu_f = f^\mu \mathscr{L}_{YM}$ when f^μ is a conformal Killing vector. Hint: use (E2.8) and (2.8b). From Noether's theorem derive the conserved current (2.34). Are the two- and three-dimensional Yang-Mills theories conformally invariant? If not, can one modify the transformation law for the vector potential, Eq. (2.26) or (2.36), to make the lower-dimensional theories conformally invariant? Examine separately the non-Abelian and Abelian models.

The transformations (2.40) and (2.41), or in finite form (2.42) and (2.43), together with the Poincaré transformations (2.28) and (2.29) form the fifteen-parameter four-dimensional conformal group, SO(4, 2), and the classical four-dimensional Yang-Mills theory is invariant against this large group of symmetry operations. A related fact is that the only parameter of the classical theory, the coupling constant g, is dimensionless, in units where $\hbar$ and the velocity of light are dimensionless.

Exercise 2.15. By expanding the finite special conformal transformation (2.43) for large c, show that up to terms of $O(c^{-2})$ it can be viewed as the following sequence of transformations: translation, improper Lorentz transformation, dilatation and coordinate inversion

$$x^\mu \to x'^\mu = x^\mu / x^2 \quad . \tag{E2.11}$$

This transformation cannot be constructed in infinitesimal form; but any conformally invariant theory is also inversion invariant. Show further that any finite special conformal transformation may be written as an inversion, followed by a translation, and then followed by another inversion.

Exercise 2.16. Show that the Lie bracket (E2.6) of two conformal Killing vectors is again a conformal Killing vector. Verify that Lie bracketing the various infinitesimal transformations of the conformal group reproduces the Lie algebra of the conformal groups; see ref [6] for the structure of this algebra.

The classical conformal symmetry cannot be maintained in the quantized version of the theory. Operationally this means that the renormalized energy-momentum tensor possesses a non-vanishing trace and there are anomalies in the dilatation and conformal currents: rather than the vanishing of (2.38), we get [10]

$$\partial_\mu j_f^\mu = \tfrac{1}{4}\partial_\alpha f^\alpha \theta^\mu_\mu \quad ; \tag{2.44}$$

since $\frac{1}{4}\partial_\alpha f^\alpha$ is nonzero for conformal Killing vectors, we see that the non-vanishing trace of $\theta^{\mu\nu}$ spoils the conformal symmetry.

The study of this anomalous breaking of conformal symmetry is of enormous practical significance in application to high-energy particle physics. If the symmetry were not broken, it is difficult to see how the theory could avoid being trivial, since it is unlikely that non-trivial scattering amplitudes can be constructed with only one dimensionless parameter. The anomalous response of a theory to conformal, more specifically scale, transformations, is the subject of the renormalization group [11], but neither this nor the trace anomalies involve any topological ideas, except when coupling to gravity is included. In flat space, the anomalous trace of the energy-momentum tensor is proportional to the Lagrangian, where the proportionality is established with the help of the renormalization group Gell-Mann-Low function [12]. Only in curved space do topologically interesting structures contribute to θ^μ_μ [13].

I shall, therefore, not elaborate on the subject of trace anomalies any further, beyond observing that this anomalous symmetry breaking is not unexpected in a quantum field theory. Conformal symmetry requires that there be no dimensional parameters. But a renormalization procedure necessarily introduces a dimensional quantity: the scale at which the theory is renormalized, and this breaks the scale symmetry and hence the conformal symmetry. More technically, field theoretical calculations must be regularized to avoid infinities and regularization is effected by introducing dimensional, conformal symmetry violating cutoff parameters, or by analytic continuation of the dimensionality of space-time, which also violates conformal symmetry, since the trace of $\theta_{\mu\nu}$ vanishes only in four dimensions.

In subsequent lectures, I shall discuss anomalies in great detail, in examples involving fermionic axial vector currents. These anomalies possess topologically non-trivial characteristics [14].

2.3 Couplings to Matter Fields

The pure Yang-Mills theory contains only vector gauge potentials. It is of interest to introduce couplings to other fields, the so-called matter fields. We consider multiplets of real scalar fields ϕ_n and spinor field ψ_n, which are taken to transform under a rigid (global) gauge transformation according to some definite representation of the Yang-Mills invariance group,

$$\delta\phi_n = -\tau^a_{nn'}\theta^a\phi_{n'} \qquad \delta\psi_n = -\tau^a_{nn'}\theta^a\psi_{n'} \qquad \theta^a \text{ constant} \quad . \tag{2.45}$$

Here the τ^a's are representation matrices for the Lie algebra; they need not coincide with the T^a representation matrices used earlier, nor need they be the same for the scalar and spinor fields, although they still satisfy the commutation relation (2.2).

$$[\tau^a , \tau^b] = f_{abc}\,\tau^c \quad . \tag{2.46}$$

For example, the scalar fields may transform according to the SU(N) adjoint representation,

$$\tau^a_{nn'} = f_{nan'} \quad , \tag{2.47a}$$

and the spinor fields according to the fundamental SU(N) representation,

$$\tau^a = \sigma^a/2\mathrm{i} \quad \text{for SU(2)} , \qquad \tau^a = \lambda^a/2\mathrm{i} \qquad \text{for SU(3)} , \text{ etc.} \tag{2.47b}$$

We then define a gauge covariant derivative of the matter fields by

$$(\mathscr{D}_\mu\phi)_n = \partial_\mu\phi_n + gA^a_\mu\tau^a_{nn'}\phi_{n'} \quad ,$$

$$(\mathscr{D}_\mu\psi)_n = \partial_\mu\psi_n + gA^a_\mu\tau^a_{nn'}\psi_{n'} \quad , \tag{2.48}$$

and one readily verifies that the covariant derivative responds covariantly to a local gauge transformation,

$$\delta(\mathscr{D}_\mu \phi)_n = -\tau^a_{nm}\theta^a(\mathscr{D}_\mu \phi)_m \quad ,$$
$$\delta(\mathscr{D}_\mu \psi)_n = -\tau^a_{nm}\theta^a(\mathscr{D}_\mu \psi)_m \quad . \tag{2.49}$$

Again a uniform matrix formalism may be used. In the notation that the matter multiplet (scalar or spinor) is a column vector in internal symmetry space, the covariant derivative is a matrix operator

$$\mathscr{D}_\mu = \partial_\mu + \mathscr{A}_\mu \quad , \tag{2.50a}$$

where the matrix potential $\mathscr{A}_\mu$ is now defined in the same representation as the matter field,

$$\mathscr{A}_\mu = gA^a_\mu \tau^a \quad . \tag{2.50b}$$

The matter fields transform under a finite gauge transformation $\mathscr{U}$, appropriate to their representation, according to the inverse rule,

$$\mathscr{U} = \mathrm{e}^{\tau^a\theta^a} = I + \tau^a\theta^a + \ldots,$$
$$\phi \to \phi^{\mathscr{U}} = \mathscr{U}^{-1}\phi \quad , \quad \psi \to \psi^{\mathscr{U}} = \mathscr{U}^{-1}\psi \quad , \tag{2.51}$$

while the vector potential matrix in this representation, $\mathscr{A}_\mu$, transforms as in Eq. (2.17); see Exercise 2.1.

$$\mathscr{A}_\mu \to \mathscr{A}^{\mathscr{U}}_\mu = \mathscr{U}^{-1}\mathscr{A}_\mu\mathscr{U} + \mathscr{U}^{-1}\partial_\mu\mathscr{U} \quad . \tag{2.52}$$

Consequently the transformation law for the gauge covariant derivative is

$$\mathscr{D}_\mu\phi \equiv \partial_\mu\phi + \mathscr{A}_\mu\phi \to \partial_\mu\phi^{\mathscr{U}} + \mathscr{A}^{\mathscr{U}}_\mu\phi^{\mathscr{U}}$$
$$= \mathscr{U}^{-1}\mathscr{D}_\mu\psi \quad . \tag{2.53a}$$

and similarly for the Fermi fields.

$$\mathscr{D}_\mu\psi = \partial_\mu\psi + \mathscr{A}_\mu\psi \to \partial_\mu\psi^{\mathscr{U}} + \mathscr{A}^{\mathscr{U}}_\mu\psi^{\mu}$$
$$= \mathscr{U}^{-1}\mathscr{D}_\mu\psi \quad . \tag{2.53b}$$

Exercise 2.17. Prove that

$$[\mathscr{D}_\mu, \mathscr{D}_\nu] \ldots = \mathscr{F}_{\mu\nu} \ldots, \quad \mathscr{F}_{\mu\nu} = \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu + [\mathscr{A}_\mu, \mathscr{A}_\nu] \quad . \qquad \text{(E2.12)}$$

The kinetic part of a gauge invariant matter Lagrangian for Dirac spinor and scalar fields is

$$\mathscr{L}_M = i\hbar \bar{\psi} \gamma^\mu \mathscr{D}_\mu \psi + \tfrac{1}{2}(\mathscr{D}^\mu \phi)(\mathscr{D}^\mu \phi) \quad , \qquad (2.54)$$

$$I_M = \int dx\, \mathscr{L}_M \quad , \qquad (2.54)$$

where I am using familiar Dirac theory quantities.

$$\bar{\psi} = \psi^\dagger \gamma^0 \quad , \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad , \quad \bar{\gamma}^\mu \equiv \gamma^0 (\gamma^\mu)^\dagger \gamma^0 = \gamma^\mu \quad . \qquad (2.55)$$

It is seen that local gauge invariance completely fixes the gauge potential — matter coupling. Of course there may also be further, pure matter, contributions to the Lagrangian, like a fermion mass term $m\bar{\psi}\psi$, boson mass term $\frac{1}{2}\mu^2\phi^2$, fermion-boson Yukawa couplings, and boson self-couplings governed by a potential $V(\phi)$. Provided these are invariant against rigid gauge transformations, local gauge invariance places no further constraints, and the parameters (masses, coupling constants) are arbitrary. But precisely because of their arbitrariness, they are unattractive to the theorist seeking a fundamental theory where everything is determined. Consequently, we shall not pay very much attention to these possible terms. (One can also construct locally gauge invariant matter-gauge field interaction with an arbitrary coupling constant, e.g. $\bar{\psi}\tau^\alpha \sigma^{\mu\nu} \psi F^a_{\mu\nu}$ with $\sigma^{\mu\nu} \equiv (1/2i)[\gamma^\mu, \gamma^\nu]$. This would not be renormalizable in four dimensions. Such interactions are called non-minimal and I shall not consider them.)

The field equation now becomes

$$(D_\mu F^{\mu\nu})^a = -\frac{\delta I_M}{\delta A^a_\nu} \equiv g J^\nu_a \quad , \qquad (2.56)$$

where the matter current J^μ_a for the Lagrangian (2.54) is

$$J^\mu_a = -i\hbar \bar{\psi} \gamma^\mu \tau^a \psi - (\mathscr{D}^\mu \phi)\tau^a \phi \quad . \qquad (2.57)$$

Note that with the definition (2.56) the current contains a factor of $\hbar$, because the fermion Lagrangian does. Similarly fermion Noether symmetry currents have an $\hbar$ coefficient, since the canonical momentum is proportional to $\hbar$. In addition to (2.56), there are also the matter field equations.

$$\frac{\delta I_{\mathrm{M}}}{\delta \psi} = 0 \quad , \qquad \frac{\delta I_{\mathrm{M}}}{\delta \phi} = 0 \quad . \tag{2.58}$$

In our example the fermion interactions are parity conserving and the current J^{μ}_{a} is a vector. One may also deal with parity non-conserving interactions involving γ_5 and axial vector currents. The γ_5 matrix is defined only in even dimensions, hence we consider two and four dimensions. (Three-dimensional fermions will be discussed in the last section.)

$$\text{two dimensions:} \quad \gamma_5 = \frac{1}{2\mathrm{i}} \epsilon_{\alpha\beta}\gamma^{\alpha}\gamma^{\beta} = \mathrm{i}\gamma^0\gamma^1 \quad ,$$

$$\text{four dimensions:} \quad \gamma_5 = -\frac{1}{4!} \epsilon_{\alpha\beta\gamma\delta}\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\delta} = \gamma^0\gamma^1\gamma^2\gamma^3 \quad ,$$

$$\bar{\gamma}_5 = \gamma_5 \quad , \qquad (\gamma_5)^2 = -I \quad . \tag{2.59}$$

In an even-dimensional non-Abelian theory a pure axial vector interaction with Dirac fermions would not be gauge invariant, but either of the two combinations $\mathrm{i}\bar{\psi}\gamma^{\mu}\frac{1}{2}(1 \pm \mathrm{i}\gamma_5)\tau^a\psi A^a_{\mu}$ is an allowed interaction. The chiral combinations $\frac{1}{2}(1 \pm \mathrm{i}\gamma_5)$ are projection operators, so the above may also be written in terms of Weyl spinors.

$$\psi_{\pm} = \tfrac{1}{2}(1 \pm \mathrm{i}\gamma_5)\psi \quad , \qquad \mathrm{i}\gamma_5\psi_{\pm} = \pm\psi_{\pm} \quad ; \tag{2.60a}$$

$$\bar{\psi}_{\pm} = \bar{\psi}\tfrac{1}{2}(1 \mp \mathrm{i}\gamma_5) \quad , \qquad -\bar{\psi}_{\pm}\mathrm{i}\gamma_5 = \pm\bar{\psi}_{\pm} \quad ; \tag{2.60b}$$

$$\psi = \psi_{+} + \psi_{-} \quad . \tag{2.60c}$$

Since $\bar{\psi}_{\pm}\gamma^{\mu}\psi_{\mp}$ vanishes, we see that only one projection couples:

$$\bar{\psi}\gamma^{\mu}(\partial_{\mu} + \tfrac{1}{2}(1 \pm \mathrm{i}\gamma_5)\mathscr{A}_{\mu})\psi = \bar{\psi}_{\pm}\gamma^{\mu}(\partial_{\mu} + \mathscr{A}_{\mu})\psi_{\pm} + \bar{\psi}_{\mp}\gamma^{\mu}\partial_{\mu}\psi_{\mp} \quad . \tag{2.61}$$

To preserve local gauge invariance, the right-handed spinor (+) must transform according to $\mathcal{U}^{-1}$ and the left-handed spinor (−) is a singlet, or *vice versa*. Also, there must be no couplings between right- and left-handed spinors; in particular there can be no fermion mass term: $\bar{\psi}_\pm \psi_\pm = 0$. In other words, the gauge group transformation (2.51) is defined separately for the right-handed Weyl spinors and for the left-handed Weyl spinors. Frequently one deletes spinors of one chirality entirely from the theory, and deals with massless Weyl spinors of definite chirality (handedness). Alternatively both may be present, transforming according to different representations of the gauge group. [If both transform by the same representation matrix, the vector theory (2.54) is regained.] Of course the field equations remain as in (2.56) and (2.58), with the current J^μ appropriately modified.

In an Abelian theory, a pure γ_5 vector coupling is allowed, but the fermions must be massless. This can be called axial spinor electrodynamics.

The matter current in a gauge theory satisfies an important constraint: it must be covariantly conserved. This follows when we differentiate covariantly the Yang-Mills equation (2.56). In matrix notation we find

$$D_\nu g J^\nu = D_\nu D_\mu F^{\mu\nu} = \tfrac{1}{2}\left[D_\nu D_\mu - D_\mu D_\nu\right] F^{\mu\nu} = \tfrac{1}{2}\left[F_{\mu\nu}, F^{\mu\nu}\right] = 0 \ . \tag{2.62}$$

Indeed, in a gauge invariant theory the field equations (2.58) and the current definition (2.56) will imply covariant conservation of J^μ. However, once again I must remind you that all these manipulations are formal; they are valid on the classical level but must be re-examined in the quantum theory. As we shall see later, there are instances, involving chiral couplings, when the source current J^μ possesses an anomalous, non-vanishing divergence. In that case there are obstacles to constructing the quantum gauge theory, which we do not know how to overcome, even though classically there is no evidence for the problem. So the absence of anomalies in source currents is an important constraint on building quantum gauge field theory models.

Exercise 2.18. With the matter Lagrangian as in (2.54), show that the field equations (2.58) imply the covariant conservation of the current (2.57). Show also that $D_\mu J^\mu = 0$ is equivalent to the statement that the matter action is invariant against infinitesimal gauge transformations. Hint: recall the definition of J^μ in (2.56).

Exercise 2.19. The combined Yang-Mills matter Lagrangian is invariant under rigid gauge transformations. Using Noether's theorem, derive the conserved symmetry current j^μ_a. Show that in addition to its pure gauge field part (2.23) it now acquires a matter contribution given by J^{μ}_{a}. Show that the ordinary conservation of j^{μ}_{a} is equivalent to the covariant conservation of J^{μ}_{a} and the gauge field equation (2.56). Finally, show that the ordinary divergence of $F^{\mu\nu}_{a}$ is given by j^{μ}_{a}; hence the conservation of the latter is also required by the anti-symmetry of the field tensor.

Of course the interaction with matter preserves Poincaré invariance and this symmetry is maintained in the quantum theory for all cases of interest. (In two dimensions, with chiral couplings a conserved gauge invariant and local energy-momentum tensor cannot be constructed in the quantum theory; also the quantized model is beset by anomalies in the source current; more about this later.)

In four dimensions and if there are no dimensional parameters in the matter Lagrangian (mass terms, cubic scalar field self-couplings), conformal symmetry holds on the classical level, but just as for pure Yang-Mills, it is anomalously broken after quantization. The scale and special conformal transformation rules for four-dimensional scalar and spinor fields are given by

Dilatation transformation $f^{\mu} = ax^{\mu}$,

$$\delta_f \phi = f^{\alpha}\partial_{\alpha}\phi + a\phi \quad ,$$

$$\delta_f \psi = f^{\alpha}\partial_{\alpha}\psi + \tfrac{3}{2}a\psi \quad ; \tag{2.63}$$

Special conformal transformation $f^{\mu} = 2c \cdot x x^{\mu} - c^{\mu}x^{2}$,

$$\delta_f \phi = f^{\alpha}\partial_{\alpha}v + 2c \cdot x\phi \quad ,$$

$$\delta_f \psi = f^{\alpha}\partial_{\alpha}\psi + (3c \cdot x + ic_{\alpha}\sigma^{\alpha\beta}x_{\beta})\psi \quad . \tag{2.64}$$

Note that these are not Lie derivatives; indeed that operator is not defined for spinor fields.

Exercise 2.20. Show that the gauge covariant coordinate transformations $\bar{\delta}_f\phi$ and $\bar{\delta}_f\psi$ are as above except that the ordinary derivatives are replaced by the appropriate covariant derivaties.

Exercise 2.21. Verify that the above field transformation rules satisfy the proper commutation relations (E2.7).

Exercise 2.22. For the above transformations, prove $\delta_f \mathcal{L}_{\mathrm{M}} = \partial_{\mu}\Omega^{\mu}_{f,\mathrm{M}}$. Note that when scalar fields are present $\Omega^{\mu}_{f,\mathrm{M}} \neq f^{\mu}\mathcal{L}_{\mathrm{M}}$. Hence, further improvement of the energy-momentum tensor is required to arrive at the current $\theta^{\mu\nu}f_{\nu}$; see ref. [6].

Finally let us comment on the role that all the fields which we have discussed play in physical theories. For strong interactions, one believes that Yang-Mills fields provide the "glue" that binds the quarks, described by Dirac spinor fields, in the hadrons. No observed particles are associated with elementary excitations of the gauge or Dirac fields [15]. These excitations are thought to be confined — a speculation which is well supported by various plausible arguments and approximate calculations, but no convincing proof has yet been given. For weak and electromagnetic interactions the vector potentials correspond to observed immediate vector bosons (W, Z) and photons that mediate these forces between quarks and leptons, which again are described by spinor fields coupling in chiral fashion to the gauge fields [16]. The W and Z are massive, yet local gauge invariance prohibits a conventional vector meson mass term in the Lagrangian — as mentioned earlier, $M^2 A_\mu^a A^{\mu a}$ is not gauge invariant. In order to circumvent this problem, model builders [17] have invoked the mechanism of spontaneous gauge symmetry breaking. One would like to see this breaking occur for dynamical reasons in a theory involving just gauge potentials and massless spinor fields, so that only one parameter — the gauge coupling constant — characterizes the theory. Unfortunately, no realistic and convincing model has been constructed in which this attractive speculation can be plausibly supported [18].

Failing at this, one arranges for the spontaneous breaking by introducing scalar fields whose gauge-invariant self-couplings are so chosen that the lower energy configuration is indeed non-symmetric; i.e., one uses the Goldstone-Higgs mechanism [17]. While there are many unattractive aspects to this procedure — the first being its *ad hoc* nature — it is phenomenologically successful [19], and this is where the subject stands today.

2.4 Classical Gauge Fields

While our aim is to discuss the quantized Yang-Mills theory, let us pause for a moment and examine the dynamical field equations in their classical setting. After all, the Maxwell theory, which is the antecedent and inspiration for Yang-Mills theory, was thoroughly investigated within classical physics, with results that are quite relevant physically even when quantum effects are ignored. Unfortunately, no such physical success can be claimed here, though much of mathematical interest has been achieved.

We consider first the sourceless equation in four dimensions,

$$D_\nu F^{\mu\nu} = 0 \quad . \tag{2.65}$$

In discussing solutions, it will be useful to characterize them by their energy, momentum and angular momentum. These expressions are of course determined by the energy-momentum tensor (2.33), and take familiar electromagnetic form in terms of the non-Abelian electric and magnetic fields.

$$E_a^i = F_{0i}^a \quad ,$$

$$B_a^i = -\tfrac{1}{2}\epsilon^{ijk}F_{jk}^a \qquad \text{(three spatial dimensions)},$$

$$B_a = -\tfrac{1}{2}\epsilon^{ij}F_{ij}^a \qquad \text{(two spatial dimensions)}. \tag{2.66}$$

(The discussion will be confined to the theory in three spatial dimensions.) The energy density is θ^{00},

$$\mathscr{E} = \tfrac{1}{2}(\boldsymbol{E}_a^2 + \boldsymbol{B}_a^2) \quad , \qquad E = \int \mathrm{d}\boldsymbol{r}\,\mathscr{E} \quad , \tag{2.67a}$$

the momentum density is the Poynting vector θ^{0i},

$$\mathscr{P} = \boldsymbol{E}_a \times \boldsymbol{B}_a \quad , \qquad \boldsymbol{P} = \int \mathrm{d}\boldsymbol{r}\,\mathscr{P} \; , \tag{2.67b}$$

and the angular momentum density $\epsilon^{ijk}x^j\theta^{0k}$ is given by

$$\mathscr{M} = \boldsymbol{r} \times (\boldsymbol{E}_a \times \boldsymbol{B}_a) \, , \qquad \boldsymbol{M} = \int \mathrm{d}\boldsymbol{r}\,\mathscr{M} \; . \tag{2.67c}$$

The first issue concerns the existence of regular solutions. If regular initial data is taken, will the solution evolve in a regular fashion, or will the nonlinearities produce singularities? This question has been answered: regular solutions to (2.65) do exist, and the same is true if one considers a larger system: scalar and spinor fields interacting with gauge fields [20].

However, physicists are not so interested in the general solution which depends on arbitrary initial data, but rather in specific solutions which reflect some physically interesting situation. For example, in the Maxwell theory we are interested in plane wave solutions.

Let us note that any Maxwell solution is a solution of the Yang-Mills equation, when one makes the *Ansatz* that the space and internal symmetry degrees of freedom decouple. If one forms $A_\mu^a(x) = \eta^a A_\mu(x)$ with η^a constant and $A_\mu(x)$ satisfying the Maxwell equation, then $A_\mu^a(x)$ is a solution to the Yang-Mills equation, which we shall call "Abelian".

It is interesting to see whether there are plane wave solutions in the non-Abelian theory, which are not Abelian. By "plane wave", we shall mean a configuration of finite energy density ($0 < \mathscr{E} < \infty$), of constant direction for the Poynting vector [$\mathscr{P}(x) = \hat{\mathscr{P}}\,|\mathscr{P}(x)|$ with $\hat{\mathscr{P}}$ constant], and with magnitude of the Poynting vector equal to the energy density ($|\mathscr{P}| = \mathscr{E}$). Such solutions have been constructed [21], but unlike their Maxwell analogs, they do not seem to have any physical significance. Certainly, if gauge quanta are confined, one cannot make a coherent superposition of them to construct an observable plane wave. Alternatively, one may view the Maxwell waves as quantum mechanical wave functions for the photon. However, the non-Abelian plane waves solve a nonlinear equation; they cannot be superposed to form other solutions, and it is hard to see how they can be used as wave functions.

Another class of solutions, more appropriate to nonlinear field theories, are the celebrated solitons, which *do* have a quantum meaning — they are the starting point of a semi-classical description of coherently bound quantum states [22]. A soliton should be a static solutions, have finite energy and be stable in the sense that small perturbations do not grow exponentially in time. However, one proves with virial theorems that no such solution exists in the pure Yang-Mills theory in four, three or two dimensions [23].

Another tack that one can take is that of symmetry. Recall that the classical Yang-Mills theory in four dimensions possesses conformal SO(4, 2) symmetry. One may seek solutions invariant under the maximal compact subgroup, i.e. SO(4) × SO(2). This solution has been constructed [24]; it is called a "meron". But again no physical significance has been attached to it, or to its generalization which possesses the smaller compact invariance symmetry group, SO(4) [25].

There are many other solutions to (2.65) that have been found [26], and while their discoverers invariably highlight some unique characteristic, no physical application has been given thus far — although doubtlessly they are mathematically interesting.

In the above discussion, I have touched on a subject which is worth elaborating upon. I spoke of the SO(4) × SO(2) invariant meron solution. But what exactly does one mean by "invariance" in a gauge theory? Let us approach this question first by considering a scalar field $\phi(x)$. We say that a given functional form for ϕ is invariant against translations in the a^μ direction if there is no dependence on $x \cdot a$, or equivalently if $a^\mu \partial_\mu \phi = 0$. Similarly, ϕ is rotationally invariant if it depends only on $|r|$ and not on angles, i.e., if $\epsilon^{ijk} r^i \partial_k \phi = 0$. In both cases we see that the derivative is a Lie derivative (2.27) with respect to an infinitesimal coordinate transformation against which the ϕ field configuration is invariant. Consequently we define an arbitrary tensor field $T^{\nu\cdots}_{\mu\cdots}$ to be invariant against an arbitrary coordinate transformation, infinitesimally given by f^{μ}, when

$$L_f T^{\nu\cdots}_{\mu\cdots} = 0 \Leftrightarrow \text{invariant tensor field.} \tag{2.68}$$

However, for a gauge potential A_μ, this definition is too restrictive since we are interested in coordinate invariance of gauge invariant quantities, and not necessarily of a gauge variant object like the potential. Therefore we extend (2.68) by saying that a given gauge potential A_μ is invariant against a coordinate transformation not only if the Lie derivative annihilates it, but more generally if the Lie derivative acting on A_μ produces an infinitesimal gauge transformation.

$$L_f A_\mu = D_\mu W_f \Leftrightarrow \text{invariant gauge potential.} \tag{2.69a}$$

(This condition refers to arbitrary gauge potential configurations, not necessarily solutions to Yang-Mills equations.) One may transform (2.69a) by recalling the earlier result that a Lie derivative of A_μ supplemented by a gauge transformation with gauge function $-f^{\alpha}A_{\alpha}$ equals $f^{\alpha}F_{\alpha\mu}$; see Eq. (2.35). Hence an equivalent test for coordinate invariance is

$$f^{\alpha}F_{\alpha\mu} = D_\mu \Phi_f \Leftrightarrow \text{invariant gauge field.} \tag{2.69b}$$

This is a gauge covariant test, because it is applied to the gauge covariant field strength and not to the gauge-variant potential.

Aside from giving a convenient criterion for deciding whether some gauge field configuration is invariant under coordinate transformations, the formalism allows for the construction of the most general invariant gauge fields: given f^{μ}, one solves (2.69a) for A_μ with W_f arbitrary [27].

Also, these ideas have been used in the following "physical" ways. Observe that Φ_f in (2.69b) is a gauge covariant Lorentz scalar field. Hence we see that an invariant gauge field always has some components that involve scalar fields. If one considers a pure gauge theory in $4 + n$ dimensions, and dimensionally reduces to four dimensions, by asserting that the potentials are independent of the n additional dimensions, i.e., that they are invariant against coordinate transformations in those n directions, then some components of the $(4 + n)$-dimensional $F_{\mu\nu}$ survive in four dimensions as scalar fields. It has been suggested that the scalar fields necessary for spontaneous symmetry breaking in the Weinberg-Salam model [17] might arise in this fashion from dimensional reduction of a higher-dimensional pure Yang-Mills model [28]. However, as with all attempts to replace the *ad hoc* symmetry-breaking procedures based on scalar fields with something more natural (is dimensional reduction "natural"?), this idea, though promising at first, has not reproduced all the phenomenologically necessary details.

Another observation is interesting: the quantity Φ_f, defined in (2.69b), has physical significance. Consider a particle moving freely in space — there will be constants of motion C_f associated with that motion as a consequence of the (free) dynamics being invariant against coordinate transformations f^μ, e.g. energy and momentum as a consequence of time and space translation invariance, angular momentum as a consequence of rotational invariance, etc. Now consider the same particle moving in some background gauge field. In general, an arbitrary background field will break the invariance and the constants of motion will disappear. However if the background is itself invariant, the constants of motion will remain, but their form will be modified by a contribution from the field: a term proportional to Φ_f must be added to C_f. This explains the frequently noted fact that particles moving in prescribed gauge fields have unexpected contributions to their constants of motion [29].

Exercise 2.23. The above remarks apply to an Abelian theory as well. Consider a charged particle moving non-relativistically in a magnetic monopole field $\boldsymbol{B} = g\hat{\boldsymbol{r}}/r^2$ according to the Lorentz force law,

$$m\ddot{\boldsymbol{r}} = e\dot{\boldsymbol{r}} \times \boldsymbol{B} \quad . \tag{E2.13}$$

Show that the conserved angular momentum $\boldsymbol{J}$ has an unexpected contribution beyond $\boldsymbol{r} \times m\dot{\boldsymbol{r}}$,

$$\boldsymbol{J} = \boldsymbol{r} \times m\dot{\boldsymbol{r}} - eg\hat{\boldsymbol{r}} \quad . \tag{E2.14}$$

Next consider an infinitesimal rotation $f^i = \epsilon^{ijk} r^j \omega^k$ (ω is the rotation parameter). Compute $f^i F_{ij}$ with the above magnetic field and show that it is a gradient of a scalar Φ. Compare with (E2.14). (Note: here g is the monopole strength, not the gauge theory coupling constant.)

Returning now to classical solutions, let us take note of another category that has been studied: Yang-Mills fields with prescribed external delta-function sources, i.e., solutions to (2.65) that are singular, with the singularity giving rise to a delta function which is interpreted as a localized source. Again all sorts of interesting phenomena have been found, but physical relevance is obscure, presumably because it is not sensible to approximate quark sources (which should be confined) by a classical localized source [30]. I mention briefly some salient results:

(1) The Coulomb potential, being a Maxwell solution, also solves the Yang-Mills equation with a delta-function point source. However, when

the source strength exceeds a critical magnitude, the solution becomes unstable [31].

(2) The same source can produce more than one, gauge non-equivalent, solution as the source strength is increased beyond a critical value, i.e. there is a bifurcation phenomenon [32, 33].

(3) Some static solutions are stable, without minimizing the energy. They are stabilized by gyroscopic forces analogous to those operating in a spinning top [33, 34].

(4) The Dirac magnetic point monopole, which solves the Abelian theory, but with a singular vector potential containing "strings", may be represented in a non-Abelian theory by a solution which is regular (aside from the singularity at the location of the monopole). For example in SU(2), the following describes a point monopole at the origin [35]:

$$A_a^0 = 0, \qquad A_a^i = \frac{1}{g}\,\epsilon^{aij}\frac{\hat{r}^j}{r} \quad . \tag{2.70a}$$

The above solves (2.65) and is SU(2) gauge equivalent to

$$A_a^0 = 0\ , \quad A_{1,2} = 0\ , \quad A_3 = A_{\text{Dirac}}\ , \qquad \nabla\times A_{\text{Dirac}} = \frac{1}{g}\frac{\hat{r}}{r^2}\quad . \tag{2.70b}$$

Note that the potential in (2.70a) is not manifestly rotationally symmetric since it mixes spatial degrees of freedom with internal degrees of freedom — but it is rotationally symmetric according to the criterion (2.69), as it should be since a point magnetic monopole is a rotationally symmetric object.

Exercise 2.24. Find an explicit formula for A_{Dirac}; i.e., solve the equation

$$\nabla\times A_{\text{Dirac}} = g\,\frac{\hat{r}}{r^2}\quad . \tag{E2.15}$$

Hint: Use radial coordinates. Find a gauge transformation that transform (2.70a) to (2.70b). Verify that A_{Dirac} as well as the configuration in (2.70a) are rotationally invariant in the sense (2.69a). (Note: here g is the monopole strength, not the gauge theory coupling constant.)

Example (4) leads to the final and most important category of solutions I shall mention. Now dynamical sources J^μ are included and the source current is given by scalar fields as in (2.57), while the scalar fields satisfy their own dynamical equation as in (2.58). It has been understood that whenever the gauge group is simple and the dynamics of the scalar potential is such that the symmetry group is spontaneously broken to one with a U(1) factor, a *smooth* monopole solution can exist, i.e., the solution (2.70a) acquires a form factor, which vanishes at the origin and approaches unity at large distances. Thus the configuration is regular at the origin, but far away from the origin it still appears as a monopole. Also one finds a smooth, self-consistent solution for the scalar field. Everything is static; the classical energy is finite; there is good reason to believe that the solution is stable — one is speaking of a soliton in three spatial dimensions and consequently of a true quantum state which is being described semi-classically [36].

Exercise 2.25. Call $a(r)$ the form factor occurring in the monopole soliton; i.e., consider the configuration (2.70a) with a further factor $a(r)$. Perform the gauge transformation determined in Exercise 2.24 and establish how (2.70b) is changed.

After the initial discovery of this smooth monopole solution in a SU(2) gauge theory with scalar field in the adjoint representation breaking the symmetry to U(1) — the original 't Hooft-Polyakov monopole [37] — there has been much study of this curious structure [38]. Multi-monopole generalizations and monopoles with electric charge — dyons — have been found. In addition to the obvious mathematical fascination, there is physical interest as well because current speculative "Grand Unified Theories" (which one hopes will unify the already unified weak and electromagnetic interactions with the strong interaction) are precisely of the type that support monopoles. They are based on a simple group (that is why they are unified); the symmetry must break spontaneously (Nature doesn't exhibit this unity); a U(1) factor must survive (electromagnetism exists). Interest was further spurred by the reported observation of a monopole [39].

Nevertheless no definite physical role has yet been found for monopoles and dyons in our present understanding of Nature. Conclusive experimental evidence for their existence is lacking, and their theoretical implications are problematical: cosmological models cannot easily accomodate them [40]; it appears that monopoles cause proton decay [41], yet the proton appears to be stable. Thus we do not know at the present time whether Grand Unified Theory is wrong, or whether its consequences are being improperly applied, or whether new experiments will bring everything into line. Suffice it to say, that monopoles remain important, if only in a negative way, by providing important constraints on model building.

Although particle theory has not yet absorbed the monopole soliton into a consistent phenomenology, solitons in other branches of physics have led to important new insights, especially in lower-dimensional systems that are realized in condensed matter, for example vortices in superconductors and soliton-induced charge fractionization in one-dimensional polymers, like polyacetylene [42].

While analysis of the monopole and other (lower dimensional) soliton solutions uses topological methods, this subject concerns classical field theory, and I refer you to the literature for further discussion [38].

There is one more class of solutions, which I shall describe later. These do not solve the Yang-Mills equations (2.65) in Minkowski space, but rather in Euclidean space, and are called instantons (pseudoparticles). In fact instantons solve the Euclidean self-duality equation

$$ {}^{*}F^{\mu\nu} = \pm F^{\mu\nu} \quad , \tag{2.71} $$

and then (the Euclidean-space analog of) (2.65) follows by the Bianchi identity. [In Minkowski space, the proportionality constant in (2.71) must be $\pm i$.] Of all the solutions, the instantons have interested mathematicians most; for physicists they give a semi-classical understanding of some of the topological effects that are present in Yang-Mills theory. This will be explained in a subsequent lecture.

3. Quantization

I now come to the problem of quantizing our Yang-Mills theory. It is important to develop this subject carefully, because as we shall see, the topological subtleties of theory can be uncovered in the quantization process. Since Yang-Mills theory is gauge invariant, we expect that there will be complications with a straightforward approach to the canonical formalism, as there already are in Maxwell theory. It turns out that because of local gauge invariance, we are dealing with a constrained canonical system, and therefore I shall first exemplify and solve a constrained quantum mechanical problem, which we can all understand easily.

3.1 Constrained Quantum Mechanical Example

Consider a two-body mechanical system, governed by a Hamiltonian for one-dimensional motion.

$$ H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(q_1 - q_2) \quad . \tag{3.1} $$

Obviously, the total momentum is a constant of motion, as a consequence of the translational invariance of the interaction.

$$P = p_1 + p_2 \quad , \quad \frac{i}{\hbar}\,[H, P] = 0 \quad . \tag{3.2}$$

It is useful therefore to pass to center-of-mass coordinates, which are also canonical.

$$Q = \frac{m_1 q_1 + m_2 q_2}{m_1 + m_2} \quad , \quad q = q_1 - q_2 \quad , \quad p = \frac{m_2 p_1 - m_1 p_2}{m_1 + m_2} \tag{3.3}$$

and the Hamiltonian in terms of the new canonical variables reads

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} + V(q) \equiv \frac{P^2}{2M} + H_{\mathrm{CM}}(p, q) \quad ,$$

$$M = m_1 + m_2 \quad , \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad . \tag{3.4}$$

It is now clear that H does not depend on Q — that is why P commutes with it — so H and P can be simultaneously diagonalized. The states are of the form

$$\Psi(Q, q) = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{\mathrm{i}KQ}\, \psi(q) \quad , \tag{3.5}$$

where $\hbar K$ is an eigenvalue of P and $\psi(q)$ is an eigenfunction of H_{CM}.

Suppose now that for some reason one is instructed to append to our theory the requirement that "physical" states have zero total momentum. One cannot satisfy this requirement by setting the operator $P = p_1 + p_2$ to zero; we cannot have $p_1 = -p_2$ since this would violate the commutation relations satisfied by p_i and q_i. However, we can enforce the requirement by demanding that P acting on physical states is zero.

$$P\Psi_{\mathrm{phys}} = 0 \quad . \tag{3.6}$$

This means that Ψ_{phys} does not depend on the variable conjugate to P — the same variable which is absent from H — and the general solution to (3.6) is an arbitrary function depending only on q, $\Psi_{\mathrm{phys}} = \psi(q)$. This wavefunction is governed by the Hamiltonian H_{CM}, and we can say that the constraint has been solved.

The one disadvantage of the procedure is that physical states are no longer normalizable on the full Hilbert space; clearly the integral $\int dQ dq(\Psi^*\Psi)_{\text{phys}}$ will be infinite since Ψ_{phys} does not depend on Q. This reflects the physical fact that our constraint insures that the probability of the total momentum vanishing is 1, and correspondingly the probability of finding nonzero total momentum vanishes. The solution to this difficulty is trivial: do not normalize with respect to Q. Yet one must guard against a formal contradiction: consider the expectation value of the canonical P, Q commutator between physical states,

$$\langle \text{physical} | [Q, P] | \text{physical} \rangle = i\hbar \langle \text{physical} | \text{physical} \rangle \quad .$$

The left-hand side vanishes since P annihilates physical states, while the right-hand side does not. But the contradiction disappears when one remembers that it is illegitimate to take expectation values between non-normalizable states.

It may however not be clear how to solve the constraints, and one may wish to use a functional integral involving all the variables. If there were no constraints, the integral would take the form

$$\langle x_i'; t' | x_i; t \rangle = \int \mathscr{D}p(\tau) \mathscr{D}q_i(\tau) \exp \frac{i}{\hbar} \int_t^{t'} d\tau \, [p_i \dot{q}_i - H] \ , \tag{3.7}$$

where the functional integration is over all $p_i(\tau)$ and all $q_i(\tau)$ such that $q_i(t) = x_i$ and $q_i(t') = x_i'$. To enforce the constraint, one must insert a functional delta function of $p_1(\tau) + p_2(\tau) \equiv P(\tau)$. But recall that the functional integral (without constraints) is derived by breaking up the time interval $t' - t$ into small steps of size Δt, and also representing the propagation kernel $\langle x_i'; t' | x_i; t \rangle$ by multiple ordinary integrals,

$$\langle x_i'; t' | x_i; t \rangle =$$

$$\int d\bar{p}_1 \, d\bar{x}_1 \ldots \langle x_i'; t' | \bar{p}_1; t' - \Delta t \rangle \langle \bar{p}_1; t' - \Delta t | \bar{x}_1; t' - 2\Delta t \rangle$$
$$\times \ \langle \bar{x}_1; t' - 2\Delta t | \cdots | x_i; t \rangle \quad . \tag{3.8}$$

To be sure, inserting momentum delta functions will enforce the constraint, but the $\bar{x}$ integrals will diverge, since the amplitudes do not depend on the center-of-mass coordinate. This can be remedied by inserting another functional delta function in the variable conjugate to the constraint, setting it to any arbitrary value Q_0. This is legitimate, since physical, zero-momentum states do not depend on Q. So the functional integral for the constrained problem reads

$$\langle x_i'; t' | x_i; t \rangle = \int \mathscr{D}p_i(\tau)\, \mathscr{D}q_i(\tau) \delta(P) \delta(Q - Q_0) \exp \frac{i}{\hbar} \int_t^{t'} d\tau [p_i \dot{q}_i - H] \quad . \tag{3.9}$$

Finally, if it is not clear how to identify the canonical variable Q, we may use a delta function of an arbitrary function of Q and q, provided we recall the formula

$$\delta(Q - Q_0) = \delta(f(Q, q)) \frac{\partial}{\partial Q} f(Q, q) \quad , \tag{3.10a}$$

which functionally is promoted to

$$\delta(Q(\tau) - Q_0) = \delta(f(Q(\tau), q(\tau))) \det \frac{\delta f(Q(\tau), q(\tau))}{\delta Q(\tau')} \quad . \tag{3.10b}$$

[We assume that f vanishes only at Q_0, and $(\partial/\partial Q)f$ is non-vanishing and positive there.] When it is recognized that $\delta f(Q(\tau), q(\tau))/\delta Q(\tau')$ is also the Poisson bracket between the constraint and f, we arrive, heuristically, at Faddeev's formula for the functional integral appropriate to a constrained quantum system [43]:

$$\langle x_i'; t' | x_i; t \rangle = \int \mathscr{D}p_i(\tau)\, \mathscr{D}q_i(\tau) \delta(p_1 + p_2) \delta(f) \times \det \{p_1 + p_2, f\} \exp \frac{i}{\hbar} \int_t^{t'} d\tau [p_i \dot{q}_i - H] \quad . \tag{3.11}$$

The first delta function enforces the constraint; the second involves an arbitrary function, hence it is called "a choice of gauge". The exponent is recognize to be the classical action, in terms of canonical variables.

Exercise 3.1. In the above quantum mechanical example the constraint that the total moment vanish is imposed "by hand"; it did not arise from a gauge principle. One may also construct a gauge invariant quantum mechanical example. Consider the Lagrangian

$$L = \frac{1}{2}m_1(\dot{q}_1 + eA)^2 + \frac{1}{2}m_2(\dot{q}_2 + eA)^2 - V(q_1 - q_2) \quad . \tag{E3.1}$$

Show that L is invariant under the time-dependent translation $\delta q_i(t) = a(t)$, provided the "gauge potential" $A(t)$ is also transformed. Verify that "Gauss' law" (the equation obtained by varying A) enforces the constraint, and that in the "Weyl gauge" ($A = 0$) the dynamical equations reduce to those of the above example. Show that the Lagrangian (E3.1) may be derived from (3.11), when the constraint delta function is represented by an exponential integral over A.

Exercise 3.2. Show that the Lagrangian describing motion of a point particle in a plane,

$$L = \frac{1}{2}m(\dot{r}^i + eA\epsilon^{ij}r^j)^2 - V(r) \quad , \qquad i = 1, 2 \quad , \tag{E3.2}$$

is gauge invariant against time-dependent rotations, $\delta r^i = \epsilon^{ij}r^j\omega(t)$, provided the "gauge potential" A is gauge transformed. Verify that "Gauss' law" enforces the vanishing of the angular momentum. What are the dynamical equations in the "Weyl gauge"? Solve the constraint and derive the unconstrained Hamiltonian.

3.2 Quantizing a Yang-Mills Theory

Let us now turn to the Yang-Mills theory, which is governed by the Lagrangian $\mathscr{L}_{\mathrm{YM}}$. (Since a gauge theory is trivial in two dimensions, we take the dimensionality to be three or greater. We shall use a notation appropriate to the four-dimensional theory, but an identical development can be given in any number of dimensions, with the modification that the magnetic field is not a space vector, but an anti-symmetric tensor, except in two spatial dimensions, where it is a scalar.)

$$\mathscr{L}_{\mathrm{YM}} = \tfrac{1}{2}(E_a^2 - B_a^2) \quad ,$$

$$E_a = -A_a - \nabla A_a^0 - g f_{abc} A_b^0 A_c \quad ,$$

$$B_a = \nabla_a \times A_a - \tfrac{1}{2} g f_{abc} A_b \times A_c \quad . \tag{3.12}$$

Our task is to build a Hamiltonian scheme, which will give rise to the Yang-Mills equations. These I record one again in non-covariant form. The time component of (2.16a) — the non-Abelian Gauss' law — reads

$$(D \cdot E)_a = 0 \quad , \tag{3.13}$$

and the space component — the non-Abelian Ampère law — is

$$(D^0 E)_a = (D \times B)_a \quad . \tag{3.14}$$

(In the space vector notation, the covariant derivative D has components D_i.)

The first problem that is encountered in passing to a Hamiltonian description arises from the fact that $\mathscr{L}_{\mathrm{YM}}$ does not depend on $\dot{A}_a^0$; thus there is no momentum conjugate to A_a^0. To remedy this, we use our gauge freedom to set A_a^0 to zero. This choice of gauge is called the "Weyl gauge" because Weyl publicized its use it in electrodynamics [44]. Then (3.12)-(3.14) reduce to

$$\mathscr{L}_{\mathrm{YM}} = \tfrac{1}{2}(\dot{A}_a^2 - B_a^2) \quad , \qquad E_a = -\dot{A}_a \quad ,$$

$$B_a = \nabla \times A_a - \tfrac{1}{2} g f_{abc} A_b \times A_c \quad , \tag{3.15}$$

and

$$(D \cdot E)_a = 0 \quad , \tag{3.16}$$

$$\dot{E}_a = (D \times B)_a \quad . \tag{3.17}$$

Now the Lagrangian lends itself to a canonical transcription into a Hamiltonian. The dynamical variable is A_a, its canonical momentum is $\dot{A}_a = -E_a$, and the Hamiltonian becomes

$$H = \frac{1}{2} \int \mathrm{d}r (E_a^2 + B_a^2) \quad , \tag{3.18}$$

which is also the total energy; see (2.67a). The non-vanishing canonical equal time commutator

$$[E_a^i(r), A_b^j(r')] = i\hbar \delta_{ab} \delta^{ij} \delta(r - r') \quad , \tag{3.19}$$

implies

$$\dot{A}_a = \frac{i}{\hbar}[H, A_a] = -E_a \quad , \tag{3.20a}$$

$$\dot{E}_a = \frac{i}{\hbar}[H, E_a] = (D \times B)_a \quad , \tag{3.20b}$$

and we see that the Hamiltonian equations reproduce Ampère's law (3.17) and the definition E_a in terms of $\dot{A}_a$. However, Gauss' law (3.16) has as yet not emerged, because it is a fixed-time constraint between canonical variables. (Since we are developing a fixed-time Schrödinger picture for the quantum field theory, the time argument of the operators is suppressed.)

Let us for the moment ignore the absence of Gauss' law, and observe that we have arrived at a completely consistent quantum field which, however, does not yet coincide with the Lorentz-invariant Yang-Mills theory, since we do not have Gauss' law. Certainly we cannot simply set $(D \cdot E)_a$ to zero; this operator does not commute with the canonical variables.

We observe that the Lagrangian (3.15) possesses a Noether symmetry in which A_a changes infinitesimally according to

$$\delta A_a = -\frac{1}{g}(D\theta)_a \quad , \tag{3.21}$$

where θ^a is a c-number, space dependent but time independent, function. Of course this is recognized as the residual local gauge invariance in the Weyl gauge: $A_a^0 = 0$ is preserved by time-independent gauge transformations. But now we view it as an ordinary continuous symmetry, and Noether's theorem gives the conserved charge:

$$Q_\theta = \int \mathrm{d}r\, \Pi_a \cdot \delta A_a = \frac{1}{g}\int \mathrm{d}r E_a \cdot (D\theta)_a \quad . \tag{3.22}$$

Since θ^a is arbitrary, we also know that $(1/g) \int \mathrm{d}r (D \cdot E)_a \, \theta^a$ is conserved, and so also is $(D \cdot E)_a$; a fact which may be verified by an explicit commutation with H. Note that $(D \cdot E)_a$ is not zero, since Gauss' law is not one of our operator equations. Thus we recognize that $(D \cdot E)_a$ is the time-dependent generator of infinitesimal time-independent gauge transformations, with the space-dependent c-number parameter θ^a stripped away; we call it G_a.

$$G_a = -\frac{1}{g}\,(D \cdot E)_a \quad . \tag{3.23}$$

It satisfies the following commutation relations:

$$\frac{i}{\hbar}\,[G_a(r), A_b(r')] \;=\; \frac{1}{g}\,\delta_{ab}\nabla\delta(r-r') + f_{abc}A_c(r)\delta(r-r') \;,$$

$$\frac{i}{\hbar}[G_a(r), E_b(r')] = f_{abc}E_c(r)\delta(r-r') \quad . \tag{3.24}$$

These show that G_a does indeed generate infinitesimal gauge transformations. The Hamiltonian is gauge invariant and G_a commutes with it, so G_a is time independent,

$$\dot{G}_a = \frac{i}{\hbar}\,[H, G_a] = 0 \quad . \tag{3.25}$$

The commutators of different G_a follow the Lie algebra of the gauge group,

$$\frac{i}{\hbar}[G_a(r), G_b(r')] = f_{abc}G_c(r)\delta(r-r') \quad . \tag{3.26}$$

We now see how to impose Gauss' law: the operator G_a is *not* set to zero; rather one demands that physical states be annihilated by it.

$$G_a(r)\,|\text{physical}\rangle = 0 \quad . \tag{3.27}$$

One may consider the states to be realized in a Schrödinger representation as functionals of A_a. Thus (3.27) becomes a functional differential equation satisfied by physical state functional $\Psi_{\text{phys}}(A)$,

$$\left(D \cdot \frac{\delta}{\delta A}\right)_a \Psi_{\text{phys}}(A)$$

$$= (\nabla\delta_{ab} - g f_{acb} A_c) \cdot \frac{\delta}{\delta A_b}\;\Psi_{\text{phys}}(A) = 0 \;. \tag{3.28}$$

while the Hamiltonian eigenvalue equation reads

$$\int \mathrm{d}r \left\{ -\frac{\hbar^2}{2} \frac{\delta^2}{\delta A_a^2} + \frac{1}{2} B_a^2 \right\} \Psi(A) = E \Psi(A) \quad . \tag{3.29}$$

Eqs. (3.28) and (3.29) correspond to the Yang-Mills quantum field theory, where only those solutions of the latter which also satisfy the former are physical. Equation (3.26) shows that the constraints close on commutation; hence (3.28) is integrable, at least locally.

We shall still need to examine the gauge transformation properties of our theory more closely, but let us postpone this and first discuss the constraint (3.28). The conservation of G_a, i.e., the fact that it commutes with H, means that H does not depend on certain combinations of the dynamical variables A_a. Correspondingly the constraint $G_a \Psi_{\text{phys}}(A) = 0$ forces the state functional to be independent of these quantities. As a consequence, the physical states are not normalizable, and one should guard against contradictory statements that would arise if expectation values of the commutators in (3.24) are taken between physical states.

One may proceed by solving the constraint (3.28), i.e., by finding the most general functional satisfying Gauss' law, and then deriving the effective Schrödinger equation for the unconstrained functional. For the Abelian theory this is trivial to do, since one can immediately identify the variable conjugate to $\nabla \cdot \boldsymbol{E}$; it is essentially the longitudinal component of A. Hence taking the wave functional to depend only on the transverse, but not on the longitudinal, components of A solves the constraint. This is equivalent to setting $\nabla \cdot A$ to zero, and the conventional electrodynamic Coulomb gauge emerges naturally with our approach to the Maxwell theory. Details are in the Exercises.

Exercise 3.3. Show that the most general solution to the Abelian version (3.28) in three spatial dimensions is a functional that depends on the transverse components of A, but not on the longitudinal ones. Show that a solution to the Abelian version of (3.29) is the functional

$$\Psi_0(A) = \exp\left[-\frac{1}{2\hbar} \int \mathrm{d}r\, \mathrm{d}r'\, A^i(r)\, G^{ij}(r, r')\, A^j(r') \right] \quad . \tag{E3.3}$$

with

$$G^{ij}(r, r') = (-\nabla^2 \delta^{ij} + \partial_i \partial_j) \int \frac{\mathrm{d}k}{(2\pi)^3}\, e^{ik \cdot (r - r')}\, \frac{1}{k}$$

$$= -\frac{2}{\pi^2 R^4} (\delta^{ij} - 2\hat{R}^i \hat{R}^j) \quad , \qquad R = r - r' \quad . \tag{E3.4}$$

The singularity at $R = 0$ is treated with the principal-value prescription. The energy eigenvalue

$$E = \hbar \int dr \int \frac{dk}{(2\pi)^3} k \quad , \tag{E3.5}$$

is the conventional infinite vacuum energy, hence (E3.3) is the ground state wave functional. Note that the constraint is automatically satisfied, and that (E3.3) may also be written as a functional of gauge invariant quantities,

$$\Psi_0(A) = \exp\left[-\frac{1}{4\pi^2\hbar}\int dr dr' \frac{B(r)\cdot B(r')}{|r-r'|^2}\right]. \tag{E3.6}$$

Observe that $\Psi_0(A)$ does not depend on the longitudinal component of A, hence it cannot be functionally integrated over all components of A. What is the wave functional for the one-photon state with momentum p, and what is its energy eigenvalue? What is the two-photon wave functional?

Exercise 3.4. In the Maxwell theory with an external, static, c-number charge density $\rho(r)$, Gauss' law reads

$$\nabla \cdot E = \rho \quad . \tag{E3.7}$$

Solve the constraint and show that physical states involve an arbitrary functional of the transverse part of A, times a phase factor depending on ρ and on the longitudinal part of A. Show that in the effective Schrödinger equation for the transverse functional, the energy eigenvalue includes the Coulomb energy.

For the non-Abelian theory the task is more difficult; in particular $\nabla \cdot A_a$ is *not* conjugate to $\nabla \cdot E_a$ and the Coulomb gauge does not arise naturally. In fact the constraint has been solved [45], but the effective Hamiltonian is complicated and does not lend itself to a power series expansion in g, since $1/g$ terms are present. One may understand these inverse powers of the coupling constant by recognizing that the solution of the constraint treats the non-Abelian gauge group exactly, while in the limit $g = 0$ the Yang-Mills theory is no longer invariant under a non-Abelian group of transformations. Hence, removing the non-Abelian gauge degrees of freedom exactly, which is what solving the constraint equation amounts to, prevents one from taking the limit $g = 0$.

While it would be useful to understand the dynamics of the unconstrained Hamiltonian, we have not yet succeeded in doing so. Consequently one remains with the constrained formalism and uses, for example, a path integral formulation.

Before deriving the functional integral representation of the Yang-Mills quantum theory, we re-examine the gauge transformations of the theory.