QUANTUM FIELD THEORY I
PROBLEMS
2nd semester 2022-2023

Solutions to Srendicki's problems are available online, e.g. at
`https://vdocuments.mx/qft-srednicki-solutions.html`

I: Classical field theory

(1) Consider a system of N coupled harmonic oscillators with potential

$$V = \sum_{i=1}^{N} \frac{\kappa}{2}(q_{i+1} - q_i)^2. \tag{1}$$

Determine the normal coordinates and the eigenvectors of the potential, with boundary conditions $q_0(t) = q_{N+1}(t) = 0$. Determine directly the equations of motion and the normal coordinates in the continuum limit without using the Lagrangian, by taking the continuum limit of the result obtained in the discrete case.
Solution: e.g. Sect. 2.3 of the textbook by David Morin:
`https://scholar.harvard.edu/david-morin/waves.`

(2) Consider the Lagrangian density

$$\mathcal{L}(x) = \partial_\mu \phi^* \partial_\mu \phi - m^2 |\phi|^2. \tag{2}$$

where ϕ is a complex classical field. Derive the classical equations of motion and solve them using the method of normal coordinates.

(3) *Optional*: problems 5.2a-d, 5.4 and 5.5 of Radovanović

II: The Poincaré group, symmetries and Noether's theorem

(4) Show that requiring invariance of the metric upon Lorentz transformations

$$\Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma g^{\nu\sigma} = g^{\mu\rho}, \tag{3}$$

and writing the infinitesimal transformation in terms of its generators $J_{\rho\sigma}$

$$\Lambda^\mu{}_\nu = g^\mu{}_\nu + \frac{i}{2}\omega^{\rho\sigma} J_{\rho\sigma}{}^\mu{}_\nu, \tag{4}$$

completely fixes the explicit form of the generators, viewed as 4×4 Lorenz matrices $(J_{\rho\sigma})^\mu{}_\nu$, and determine their explicit expression.
Solution: see Eq.s (13-19) of the notes on the website.

(5) Derive the Lorentz algebra, i.e. the commutation relations

$$[J^{\mu\nu}, J^{\rho\sigma}] = -i\left(g^{\mu\sigma} J^{\nu\rho} + g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho}\right) \tag{5}$$

(a) using the explicit form of the generators which was found in problem (3);

(b) from the relation $D(\Lambda')^{-1} D(\Lambda) D(\Lambda') = D(\Lambda'^{-1}\Lambda\Lambda')$, which holds for any Lorentz representation $D(\Lambda)$ in the case of infinitesimal transformations (see problem 1.11-b of Radovanović).

(6) *Optional*: problems 1.16 and 1.17 of Radovanović

(7) Show that the Lagangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_1 \partial_\mu \phi_1 + \partial_\mu \phi_2 \partial_\mu \phi_2\right) - \frac{m^2}{2}\left(\phi_1^2 + \phi_2^2\right) - \lambda \left(\phi_1^2 + \phi_2^2\right)^2 \tag{6}$$

is invariant under the transformation

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = R(\theta) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{7}$$

where $R(\theta)$ is a rotation matrix by an angle θ. Determine the Noether current and charge. *Solution: see problem 5.11 of Radovanović.*

(8) *Optional*: problems 5.12, 5.14b, 5.15, and 5.17 of Radovanović

(9) Consider the Maxwell Langrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{8}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{9}$$

Determine the energy-momentum tensor $T'^{\mu\nu}$ explicitly in terms of the fields a^μ through Noether's theorem, and show that it is not symmetric. Determine explicitly the prepotential $A^{\rho\mu\nu} = -A^{\mu\rho\nu}$ such that

$$T^{\mu\nu} = T'^{\mu\nu} + \partial_\rho A^{\rho\mu\nu} \tag{10}$$

is symmetric and given by the so-called Belinfante form

$$T^{\mu\nu} = -\frac{1}{2}\left(F^{\mu\alpha} F^{\nu}{}_{\alpha} - \mathcal{L} g^{\mu\nu}\right). \tag{11}$$

Solution: see problem 5.18 of Radovanović.

(10) *Optional*: problem 5.19 of Radovanović.

(11) Show that the momentum operator for a real scalar quantum field, namely

$$\vec{P} = -\int d^3x \dot{\phi}(\vec{x}, t) \vec{\nabla} \phi(\vec{x}, t) \tag{12}$$

generates translations of the field operator, i.e.

$$[\vec{P}, \phi(\vec{x}, t)] = -i \vec{\nabla} \phi(\vec{x}, t). \tag{13}$$

III: Canonical field quantization of scalar fields

(12) Derive the canonical commutation relation for the scalar field operator

$$[\phi(\vec{x}), \pi(\vec{x}')] = i\delta^{(3)}(\vec{x} - \vec{x}') \tag{14}$$

from the expression of the field operators ϕ and ϕ in terms of creation and annihilation operators, and the commutation relation satisfied by the latter.

(13) *Optional*: problems 7.6 b-f, 7.7 and 7.8 of Radovanović.

(14) Show that the spin operator for the electromagnetic field has the form

$$s^i = \epsilon^{ijk} \int d^3x E_j A_k \tag{15}$$

$$= \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm} \left(\epsilon^{ijk} \varepsilon^{s\,*}_{\vec{k}j} \varepsilon^{s}_{\vec{k}k} \right) a^{s\,\dagger}_{\vec{k}} a^{s}_{\vec{k}}. \tag{16}$$

(15) Prove that if γ^μ are $n \times n$ matrices satisfying the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\mu} I, \tag{17}$$

where I is the n-dimensional identity matrix, then the matrices

$$\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{18}$$

satisfy the Lorentz algebra, as given in problem 4, Eq. (4).

(16) Prove that the matrices $\sigma^{\mu\nu}$ Eq. (16) satisfy the equation

$$\Lambda^{-1}_{1/2} \gamma^\mu \Lambda_{1/2} = \Lambda^\mu{}_\nu \gamma^\nu, \tag{19}$$

where

$$\Lambda_{1/2} = \exp -\frac{i}{2} \omega_{\alpha\beta} \sigma^{\alpha\beta} \tag{20}$$

$$\Lambda^\mu{}_\nu = \left(\exp -\frac{i}{2} \omega_{\alpha\beta} (J^{\alpha\beta} \right)^\mu{}_\nu \tag{21}$$

and $J^{\alpha\beta}$ are the generators of the Lorentz group (see problem 4)

$$\left(J^{\alpha\beta}\right)^\mu{}_\nu = i \left(g^{\mu\alpha} g^\beta{}_\nu - g^{\mu\beta} g^\alpha_\nu \right). \tag{22}$$

It is sufficient to prove the result in the case of infinitesimal transformations.

(17) Write the Dirac equations satisfied by the left and right component of a generic Dirac field ψ, defined as $\psi_L \equiv \frac{\mathbb{1}-\gamma+5}{2}\psi$; $\psi_R \equiv \frac{\mathbb{1}+\gamma+5}{2}\psi$. Determine explicitly the form of the γ_5 matrix in the Weyl representation of Dirac matrices.

(18) Show that if the solutions of the Dirac equations are normalized as

$$\bar{u}_r(p) u_s(p) = 2m\delta_{rs} \tag{23}$$

then

$$u_r(p)^\dagger u_s(p) = 2E\delta_{rs}, \tag{24}$$

where $E = p^0$.

(19) *Optional*: problems 4.6, 4.10, 4.11 and 5.13 of Radovanović.

(20) Define creation and annihilation operators $b_s(p)$, $b^\dagger_s(p)$, $d_s(p)$ and $d^\dagger_s(p)$ for the Dirac field according to

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E}} \sum_s \left(u^s(p) b_s(p) e^{-ipx} + v^s(p) d^\dagger_s(p) e^{ipx} \right) \tag{25}$$

where u and v are solution to the momentum-space Dirac equation, and determine in term of these operators the expression of the Dirac Hamiltonian

$$H = \int d^3x \bar{\psi} \left(i\vec{\gamma} \cdot \vec{\nabla} + m \right) \psi(x). \tag{26}$$

Solution: see problem 8.2b of Radovanović.

(21) *Optional*: problems 5.2e, 8.2a,c 8.3 and 8.9 Radovanović.

V: Path integral and Green functions

(22) Determine the classical action for a one-dimensional free particle with fixed initial and final conditions. Determine explicitly the matrix element of the time-evolution operator

$$K(q't'; q, t) = \langle q', t'| e^{-iH(t',-t)} |q, t\rangle \tag{27}$$

and show that is equal to the exponential of the classical action, times a factor of i.
Solution: see problem 6.1b,c of Srednicki.

(23) Show that the ground-state wave functional for a free real scalar field such that

$$\hat{\phi}(x)|\phi(x)\rangle = \phi(x)|\phi(x)\rangle \tag{28}$$

has the form

$$\langle \phi(x)|0\rangle = \mathcal{N} \exp -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \phi^*(k) \sqrt{\vec{k}|^2 + m^2} \phi(k), \tag{29}$$

where $\mathcal{N}$ is a suitable normalization.
Solution: see problem 8.8 of Srednicki.

(24) Compute, using the generating functional, the four-point function

$$G^{(4)}(x_1, x_2, x_3, x_4) = \langle 0| T \left[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \right] |0\rangle \tag{30}$$

for a free real scalar field.

(25) *Optional*: problems 6.4, 6.5, 6.9, 6.10 of Radovanović.

(26) Prove that if θ_i are (complex) components of an n-dimensional vector of Grassmann numbers and B is an $n \times n$ matrix then

$$\int \prod_i d\theta_i^* d\theta_j e^{-\theta_i^* B_{ij} \theta_j} = B_{ij}^{-1} \det B. \tag{31}$$

(27) Compute explictly the two-point function for a Dirac field

$$G^{(2)}(x_1, x_2) = \langle 0| T \psi(x_1) \bar{\psi}(x_2) \rangle \tag{32}$$

by expressing the field in terms of creation and annihlation operators and using the anticommutation relations.
Solution: see problem 8.1 of Radovanović.

VI: The reduction formula and Feynman rules

(28) Prove that if one defines

$$a_{\vec{k}}(t) \equiv i \int \frac{d^3x}{\sqrt{2\omega}} e^{i(\omega t - \vec{k}\cdot\vec{x})} \frac{\overleftrightarrow{d}}{dt} \phi(\vec{x}, t) \tag{33}$$

then

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega}} \left(e^{-ikx} a_{\vec{k}}(t) + e^{ikx} a^\dagger_{\vec{k}}(t) \right). \tag{34}$$

(29) Prove the reduction formula for a fermion field, namely, for incoming and outgoing particles,

$$\langle p_1 \dots p_n | k_1 \dots k_m \rangle = \int d^4x_1 \dots d^4x_n d^4y_1 \dots d^4y_m e^{i(p_1x_1 + \dots + p_nx_n - k_1y_1 - \dots - k_my_m)} \tag{35}$$

$$\bar{u}(p_1)S^{-1}(x_1) \dots \bar{u}(p_n)S^{-1}(x_n) \langle T\left(\bar{\psi}(x_1) \dots \bar{\psi}(x_n)\psi(y_1) \dots \psi(y_m)\right)\rangle \tag{36}$$

$$S^{-1}(y_1)u(k_1) \dots S^{-1}(y_m)u(k_m), \tag{37}$$

and $u \leftrightarrow v$, $\bar{u} \leftrightarrow \bar{v}$, $k \leftrightarrow -k$ for outgoing or incoming antiparticles respectively (an outgoing particle is like an incoming antiparticle and conversely).

(30) Write down the Feynman rules for the theories defined by the following Lagrangians:

$$\mathcal{L}_1 = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - M^2\phi^2\right) + \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + g\bar{\psi}\psi\phi; \tag{38}$$

$$\mathcal{L}_2 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi; \tag{39}$$

$$\mathcal{L}_3 = \bar{\psi}_1 i\partial\!\!\!/\psi_1 + \bar{\psi}_2 i\partial\!\!\!/\psi_2 + G\bar{\psi}_1\gamma^\mu\psi_2\bar{\psi}_2\gamma^\mu\psi_1; \tag{40}$$

$$\begin{aligned}\mathcal{L}_4 = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_s^2\phi^2\right) + \\ &+ \bar{\psi}\left(i\partial\!\!\!/ - m_f\right)\psi - g'\bar{\psi}\gamma^\mu\psi B_\mu + \frac{g}{4}\phi F_{\mu\nu}F^{\mu\nu},\end{aligned} \tag{41}$$

where in Eq. (38) ψ and ϕ are respectively a Dirac field and a real scalar field (Yukawa theory); in Eq. (39) ϕ is a complex scalar field and $F^{\mu\nu}$ is the Maxwell field strength tensor (scalar electrodynamics); in Eq. (40) ψ_1 and ψ_2 are two Dirac fields (four-Fermi theory); and in Eq. (41) $F^{\mu\nu}$ is the Maxwell field strength tensor, ϕ is a real scalar field, ψ is a Dirac field and B_μ is an external vector field (i.e. such that its free Lagrangian is nor part of the Lagrangian of the given theory).

VII: Renormalization

(31) Perform one-loop renormalization of Yukawa theory, as given by the Lagrangian Eq. (38). List all superficially divergent amplitudes and the countertermes which are needed in order to make them finite, and write the renormalized Lagrangian. Compute the fermion-antifermion-scalar three-point function and use the result to determine the renormalization of the coupling.