QUANTUM MECHANICS I EXAM

03 February 2021

Answers sheet

Here we consider a one-dimensional system whose dynamics are described by the Hamiltonian

$$H = \hbar\left[\omega a^\dagger a - \mu\left(a^\dagger + a\right)\right], \tag{1}$$

where ω is a real and positive constant, μ is a real constant and a is an operator such that

$$\left[a, a^\dagger\right] = 1. \tag{2}$$

We define also the three states $|0\rangle$, $|1\rangle$ and $|2\rangle$ such that

$$a|0\rangle = 0; \qquad |1\rangle = a^\dagger|0\rangle; \qquad |2\rangle = \frac{1}{\sqrt{2}}a^\dagger|1\rangle. \tag{3}$$

(1) Here one should note that a and $a^\dagger$ satisfy the same commutation relations as the creation and annihilation operators for the one-dimensional harmonic oscillator, and the first of Eq. (3) is the same equation that defines the vacuum state of the harmonic oscillator for which a and $a^\dagger$ are creation and annihilation operators. It immediately follows that the expectation value of $a^\dagger$ in the three given states vanishes.

Thus after acting on a state $|n\rangle$ with the creation operator $a^\dagger$, it is orthogonal to it's adjoint $\langle n|$. Hence the expectation value of $a^\dagger$ in any state $|n\rangle$ is zero.

(2) Using the fact that the commutation relations Eq. (2) are the same as in the case of the creation and annihilation operators for a harmonic oscillator, we know that $N = a^\dagger a$ is the number operator, whose first three eigentstaes are those given in Eq. (3). Combining this with the conclusion from the previous exercise, we find

$$\langle n|H|n\rangle = \langle n|\hbar\left(\omega a^\dagger a - \mu\left(a^\dagger + a\right)\right)|n\rangle = \hbar\omega n \tag{4}$$

with $n = 0, 1, 2$.

(3) Let us define

$$b = a + \delta, \tag{5}$$

which allows us to rewrite the hamiltonian:

$$\begin{aligned} \frac{1}{\hbar}H &= \omega a^\dagger a - \mu\left(a^\dagger + a\right), & (6)\\ &= \omega(b^\dagger - \delta)(b - \delta) - \mu\left((b^\dagger - \delta) + (b - \delta)\right), & (7)\\ &= \omega b^\dagger b - (\omega\delta + \mu)b^\dagger - (\omega\delta + \mu)b + \omega\delta^2 + 2\mu\delta, & (8)\\ &= \omega b^\dagger b - \frac{\mu^2}{\omega}, & (9) \end{aligned}$$

where to obtain the last line, we defined

$$\delta = -\frac{\mu}{\omega}. \tag{10}$$

From this it can be seen that

$$K = -\hbar\frac{\mu^2}{\omega}. \tag{11}$$

(4) The commutation of $b^\dagger$ and b is

$$[b, b^\dagger] = [a + \delta, a^\dagger + \delta] = [a, a^\dagger] = 1. \tag{12}$$

Where it can now be seen that the Hamiltonian is of the same form, up to an additive constant, as that of the harmonic oscillator with $b^\dagger$ and b the creation and annihilation operators, respectively. This means that we can define the number operator $N = b^\dagger b$, resulting in a Hamiltonian of a well known form:

$$H = \hbar\omega N + K, \tag{13}$$

where the spectrum of eigenvalues of H is

$$\langle\bar{n}|H|\bar{n}\rangle = E_{\bar{n}} = \hbar\omega\bar{n} + K, \tag{14}$$

with $\bar{n}$ a non-negative integer. The eigenstates of H are denoted as $|\bar{n}\rangle$ in order to stress that they are not the same as the eigenstates of $a^\dagger a$. The states of Eq. (3), denoted by $|n\rangle$, are the first three eigenstates of $a^\dagger a$.

(5) Here we define the operators

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right), \tag{15}$$

$$\hat{y} = \sqrt{\frac{\hbar}{2m\omega}}\left(b + b^\dagger\right). \tag{16}$$

Which are related by

$$\hat{y} = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger + 2\delta\right) = \hat{x} + \sqrt{\frac{2\hbar}{m\omega}}\delta. \tag{17}$$

So $\hat{x}$ and $\hat{y}$ are position operators related by a fixed constant translation.

From this, one can quickly see that

$$[\hat{y}, \hat{x}] = \left[\hat{x} + \sqrt{\frac{2\hbar}{m\omega}}\delta, \hat{x}\right] = 0, \tag{18}$$

meaning $\hat{y}$ and $\hat{x}$ are compatible.

Two compatible operators have a common eigenbasis. In this case this becomes clear if we consider

$$\hat{x}|x_0\rangle = x_0|x_0\rangle, \tag{19}$$

and also

$$\hat{y}|x_0\rangle = \left(\hat{x} + \sqrt{\frac{2\hbar}{m\omega}}\delta\right)|x_0\rangle = \left(x_0 + \sqrt{\frac{2\hbar}{m\omega}}\delta\right)|x_0\rangle, \tag{20}$$

thus $|x_0\rangle$ is also an eigenstate of $\hat{y}$, but with an eigenvalue translated by a constant.

(6) If $|y_0\rangle$ is an eigenstate of $\hat{y}$ with eigenvalue y_0

$$\hat{y}|y_0\rangle = y_0|y_0\rangle, \tag{21}$$

then using Eq. (20) we see that $|y_0\rangle$ is also an eigenstate of $\hat{x}$ with eigenvalue

$$\hat{x}|y_0\rangle = \left(y_0 - \sqrt{\frac{2\hbar}{m\omega}}\delta\right)|y_0\rangle. \tag{22}$$

But two eigenstates $|x_1\rangle$, $|x_2\rangle$ of $\hat{x}$ with eigenvalues x_1, x_2 satisfy $\langle x_1|x_2\rangle = \delta(x_1 - x_2)$. So

$$\langle x_0|y_0\rangle = \delta\left(x_0 - \left(y_0 - \sqrt{\frac{2\hbar}{m\omega}}\delta\right)\right). \tag{23}$$

(7) Here we are asked to determine the expectation value of $\hat{x}$ in the state $|0\rangle$ given in Eq. (3). In this case the expectation value of x is

$$\langle 0|\hat{x}|0\rangle = \frac{\hbar}{2m\omega}\langle 0|(a+a^\dagger)|0\rangle = 0. \tag{24}$$

We are also asked to provide the expectation value in the ground state of the Hamiltonian H of Eq. (1), that is

$$\langle\bar{0}|\hat{x}|\bar{0}\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\bar{0}|(a+a^\dagger)|\bar{0}\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\bar{0}|(b+b^\dagger-2\delta)|\bar{0}\rangle = -\sqrt{\frac{2\hbar}{m\omega}}\delta \tag{25}$$

(8) The time-dependent state $|0(t)\rangle$ is

$$|0(t)\rangle = e^{-iHt/\hbar}|0\rangle. \tag{26}$$

The probability is thus

$$P = |\langle\bar{0}|0(t)\rangle|^2 = \left|\langle\bar{0}|e^{-iHt/\hbar}|0\rangle\right|^2. \tag{27}$$

But the ground state $|\bar{0}\rangle$ is an eigenstate of H, so

$$\langle\bar{0}|e^{-iHt/\hbar} = \langle\bar{0}|e^{-iE_{\bar{0}}t/\hbar}, \tag{28}$$

because H is hermitian so its left eigenstates are the same as the right eigenstates.

Thus it follows that the probability P is time-independent:

$$P = |\langle\bar{0}|0\rangle|^2. \tag{29}$$

(9) Here again the argument for time-independence that was presented in the previous exercise applies. After performing the position measurement and finding the position $x = x_0$, the corresponding wave function is a delta function:

$$\psi_{x_0}(x) = \delta(x - x_0), \tag{30}$$

where x now are eigenvalues of the position operator $\hat{x}$. In terms of the eigenvalues of $\hat{y}$ this corresponds to

$$\psi_{x_0}(y) = \delta\left(y - \left(x_0 - \sqrt{\frac{2\hbar}{m\omega}}\delta\right)\right), \tag{31}$$

where we have used Eq. (22)

The requested probability is then

$$P = |\langle\psi_{x_0}|\bar{0}\rangle|^2 = \left|\int_{-\infty}^{\infty} dx\psi_0(x)\delta\left(x - \sqrt{\frac{2\hbar}{m\omega}}\delta\right)\right|^2 = \left|\psi_{\bar{0}}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\delta\right)\right|^2, \tag{32}$$

where $\psi_0(x)$ is the standard harmonic oscillator ground state wave function given in Eq. (8.63) of the textbook.

(10) The state $|0\rangle$ is a coherent state: this can be seen by comparing our case to the property of a coherent state presented in Eq. 8.116 of the textbook (section 8.5.1):

$$b|0\rangle = (a+\delta)|0\rangle = \delta|0\rangle. \tag{33}$$

Using Eq. 8.124 from the textbook, we then find

$$|\langle\bar{0}|0\rangle|^2 = e^{-\delta^2}. \tag{34}$$

(11) The simplest way to proceed is to determine the time evolution of b:

$$\frac{db}{dt} = \frac{da}{dt} = \frac{1}{i\hbar}[a, H] = -i\left(\omega[a, a^\dagger a] - \mu[a, a^\dagger]\right) = -i\left(\omega a - \mu\right) = -i\omega b, \tag{35}$$

thus we have

$$b(t) = e^{-i\omega t} b. \tag{36}$$

This immediately determines the time evolution of a and also of $\hat{x}$:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(be^{-i\omega t} + b^\dagger e^{i\omega t} - 2\delta\right). \tag{37}$$

But the operator b annihilates the vacuum state of H, $b|\bar{0}\rangle = 0$, so only the last term contributes to the expectation value and we get for all t

$$\langle\bar{0}|\hat{x}(t)|\bar{0}\rangle = -\sqrt{\frac{2\hbar}{m\omega}}\delta. \tag{38}$$