Solution to the exam of QUANTUM PHYSICS I of 07 february 2025

(1) First write $\hat{x}$ i.t.o. $a, a^\dagger$:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \tag{1}$$

Then $\langle x \rangle = \langle \lambda | x | \lambda \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \lambda | a + a^\dagger | \lambda \rangle$. $|\lambda\rangle$ is an eigenstate of a with eigenvalue λ(coherent state), so $\langle \lambda | a | \lambda \rangle = \lambda$ and $\langle \lambda | a^\dagger | \lambda \rangle = \lambda$. Therefore we have:

$$\langle x \rangle = \lambda \sqrt{\frac{2\hbar}{m\omega}} \equiv x_0 \tag{2}$$

(2) Repeating the same for the momentum operator:

$$\hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger). \tag{3}$$

$$\langle p \rangle = \langle \lambda | p | \lambda \rangle = -i\sqrt{\frac{m\omega\hbar}{2}} \langle \lambda | a - a^\dagger | \lambda \rangle = -i\sqrt{\frac{m\omega\hbar}{2}}(\lambda - \lambda) = 0. \tag{4}$$

(3) The uncertainty in the position is defined as:

$$\Delta^2 x = \langle x^2 \rangle - \langle x \rangle^2, \tag{5}$$

so we still need $\langle x^2 \rangle$:

$$\langle x^2 \rangle = \langle \frac{\hbar}{2m\omega}(a + a^\dagger)^2 \rangle = \frac{\hbar}{2m\omega} \langle \lambda | a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a | \lambda \rangle \tag{6}$$

Since

$$[a, a^\dagger] = 1 \quad \Longrightarrow \quad aa^\dagger = 1 + a^\dagger a \tag{7}$$

we can compute the products as

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega}(4\lambda^2 + 1). \tag{8}$$

Combining this with Eq. (2), we obtain for the uncertainty

$$\Delta^2 x = \frac{\hbar}{2m\omega}(4\lambda^2 + 1) - 4\lambda^2 \frac{\hbar}{2m\omega} = \frac{\hbar}{2m\omega}. \tag{9}$$

(4) Heisenberg's EOM are

$$\frac{i}{\hbar}[H, \mathcal{O}] = \frac{d\mathcal{O}}{dt}, \tag{10}$$

and we use the H.O. Hamiltonian as given in the problem.

First, do it for x:

$$[H,x] = \frac{1}{2m}[p^2,x] + \frac{m\omega^2}{2}[x^2,x] = \frac{1}{2m}(p[p,x]+[p,x]p) = -i\hbar\frac{p}{m}. \tag{11}$$

Putting this result into Eq. (10), we get

$$\frac{dx}{dt} = \frac{i}{\hbar}(-i\hbar\frac{p}{m}) = \frac{p}{m}. \tag{12}$$

For p, we do exactly the same:

$$[H,p] = \frac{1}{2m}[p^2,p] + \frac{m\omega^2}{2}[x^2,p] = \frac{1}{2}m\omega^2(x[x,p]+[x,p]x) = i\hbar m\omega^2 x. \tag{13}$$

So the EOM for momentum looks like

$$\frac{dp}{dt} = -m\omega^2 x. \tag{14}$$

In order to solve the equations, differentiate once more. First for x:

$$\frac{d^2x}{dt^2} = \frac{1}{m}\frac{dp}{dt} = -\omega^2 x. \tag{15}$$

This is a normal 2nd order linear D.E. with standard solution of the form

$$x(t) = A\cos\omega t + B\sin\omega t, \tag{16}$$

which then implies

$$p(t) = m\omega(-A\sin\omega t + B\cos\omega t). \tag{17}$$

Imposing the boundary conditions at $t = 0$ we get

$$A = x_s; \qquad B = \frac{p_s}{m\omega}. \tag{18}$$

(5) From Eq. (4) we know that $\langle p(0)\rangle = 0$ so $B = 0$. Imposing $A = \langle x_s\rangle$ and using Eq. (2) we get

$$\langle x(t)\rangle = 2\lambda\sqrt{\frac{\hbar}{2m\omega}}\cos\omega t. \tag{19}$$

Eqquation (17) then immediately implies

$$\langle p(t)\rangle = -\lambda\sqrt{2m\omega\hbar}\sin\omega t. \tag{20}$$

(6) See pag. 157, sec. 8.5.2 of the textbook.

(7) The possible measurement outcomes are the eigenvalues of the operator. Because the operator only acts on the subspace spanned by the $|0\rangle$ and $|3\rangle$ harmonic oscillator eigenstates, the possible outcomes are the eigenvalues of the operator in this subspace, while in the rest of the Hilbert space the operator has eigenvalue 0. Hence in order to determine the probabilities of measurements whose result is nonzero the only relevant harmonic oscillator eigenstates are $|0\rangle$ and $|3\rangle$, and we can ignore the other states and focus only on the terms $c_0|0\rangle$ and $c_3|3\rangle$.

The eigenvalue condition in the subspace is

$$\begin{aligned}\mathcal{O}|\psi\rangle &= (|0\rangle\langle 3| + |3\rangle\langle 0|)(c_0|0\rangle + c_3|3\rangle) = c_3|0\rangle + c_0|3\rangle = \\ &\stackrel{!}{=} E_{\mathcal{O}}|\psi\rangle = E_{\mathcal{O}}(c_0|0\rangle + c_3|3\rangle).\end{aligned} \tag{21}$$

Then we obtain that $c_3 = E_{\mathcal{O}}c_0$ and $c_0 = E_{\mathcal{O}}c_3$, which leads to $c_0 = E_{\mathcal{O}}^2 c_0$, which tells us that $E_{\mathcal{O}} = \pm 1$. As a conclusion, the possible measurement outcomes are $E_{\mathcal{O}} = \pm 1$, and the corresponding states of the system after the measurements the corresponding (normalised) eigenstates:

$$\begin{aligned}|+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |3\rangle) \\ |-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |3\rangle).\end{aligned} \tag{22}$$

What is left is to compute the probabilities:

$$P_+ = |\langle +|\psi\rangle|^2 = \frac{1}{2}|(\langle 0| + \langle 3|)(c_0|0\rangle + c_3|3\rangle)|^2 = \frac{1}{2}|c_0 + c_3|^2. \tag{23}$$

$$P_- = |\langle +|\psi\rangle|^2 = \frac{1}{2}|(\langle 0| - \langle 3|)(c_0|0\rangle + c_3|3\rangle)|^2 = \frac{1}{2}|c_0 - c_3|^2. \tag{24}$$

(8) This is the same question as (7) if you take

$$|\lambda\rangle = \sum_{n=0}^{\infty} c_n|n\rangle. \tag{25}$$

From the previous question, the ratio of probabilities is:

$$P_+/P_- = \left|\frac{c_0 + c_3}{c_0 - c_3}\right|^2. \tag{26}$$

What remains then is to explicitly calculate the coefficients c_0 and c_3, up to an overall normalization, since we only need their ratio. We do this by plugging Eq. (25) into the equation that defines $|\lambda\rangle$:

$$a|\lambda\rangle = \lambda|\lambda\rangle \implies \sum_{n=0}^{\infty} c_n a|n\rangle = \sum_{n=0}^{\infty} c_n\lambda|n\rangle. \tag{27}$$

Using that $a|0\rangle = 0$ and $a|n\rangle = \sqrt{n}|n-1\rangle$ for $n > 0$ we have:

$$\sum_{n=0}^{\infty} c_n\lambda|n\rangle = \sum_{n=1}^{\infty} c_n\sqrt{n}|n-1\rangle = \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}|n\rangle \tag{28}$$

obtaining the condition

$$c_{n+1} = \frac{\lambda}{\sqrt{n+1}}\, c_n \implies c_n = \frac{\lambda}{\sqrt{n}}\, c_{n-1} \implies c_n = \frac{\lambda^n}{\sqrt{n!}}\, c_0 \tag{29}$$

For the probability ratio we then have, using $c_3 = \frac{\lambda^3}{\sqrt{6}} c_0$:

$$P_+/P_- = \left| \frac{c_0(1 + c_3/c_0)}{c_0(1 - c_3/c_0)} \right|^2 = \left| \frac{\sqrt{6} + \lambda^3}{\sqrt{6} - \lambda^3} \right|^2. \tag{30}$$

(9) We have two time intervals to take into account: one before and one after the measurement, respectively $t < 0$ and $t > 0$.

Before
At $t = 0$ the system is in the state $|\lambda\rangle$. In order to determine the uncertainty at all times in this state we use the results of question (5). We found that

$$\begin{gathered} p(t) = p(0)\cos\omega t - m\omega x(0)\sin\omega t \implies \\ p^2(t) = p^2(0)\cos^2\omega t + m^2\omega^2 x^2(0)\sin^2\omega t - m\omega\{x(0), p(0)\}\sin\omega t\cos\omega t, \end{gathered} \tag{31}$$

Considering now the mean values of the operators at $t = 0$ we have:

$$\begin{gathered} \langle x(0)\rangle = 2\lambda\sqrt{\frac{\hbar}{2m\omega}}; \qquad \langle p(0)\rangle = 0; \\ \langle x^2(0)\rangle = \frac{\hbar}{2m\omega}(4\lambda^2 + 1); \qquad \langle p^2(0)\rangle = \frac{m\omega\hbar}{2}\langle\lambda|a^2 + a^{\dagger 2} - aa^\dagger - a^\dagger a|\lambda\rangle = \frac{m\omega\hbar}{2}. \end{gathered} \tag{32}$$

We can substitute these into Eq. (31) to obtain:

$$\begin{aligned} \langle p(t)\rangle &= -\lambda\sqrt{2m\omega\hbar}\sin\omega t; \\ \langle p^2(t)\rangle &= \frac{m\omega\hbar}{2}\cos^2\omega t + \frac{m\omega\hbar}{2}(4\lambda^2 + 1)\sin^2\omega t. \end{aligned} \tag{33}$$

Now we can obtain $\Delta^2 p(t)$:

$$\Delta^2 p(t) = \langle p^2(t)\rangle - \langle p(t)\rangle^2 = \frac{m\omega\hbar}{2}\cos^2\omega t + \frac{m\omega\hbar}{2}\sin^2\omega t = \frac{m\omega\hbar}{2}. \tag{34}$$

We conclude that the uncertainty time-independent.

After
After the measurement, the state collapses into either $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |3\rangle)$ or $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |3\rangle)$. Noting that

$$p^2 = \frac{\hbar m\omega}{2}\left(-a^2 - {a^\dagger}^2 + 2a^\dagger a + 1\right) \tag{35}$$

and that a^2 and $-{a^\dagger}^2$ only connect oscillator eigenstates that differ by two units, and finally observing that p also has a nonvanishing matrix element only between eigenstates that differ by one unit, we get

$$\Delta^2 p = \langle p^2\rangle = \frac{\hbar m\omega}{4}\left(\langle 0|2a^\dagger a + 1|0\rangle + \langle 3|2a^\dagger a + 1|3\rangle\right) = 2\hbar m\omega. \tag{36}$$

In conclusion:

$$\Delta^2 p(t) = \begin{cases} m\omega\hbar/2 & \text{before measurement,} \\ 2m\omega\hbar & \text{after measurement.} \end{cases} \tag{37}$$

(10) Recalling the value of $\Delta^2 x$ from question (3) and the value of $\Delta^2 x$ from question (9) we immediately note that the state $|\lambda\rangle$ is a minimum uncertainty state, hence it must be a Gaussian wavepacket. Furthermore, from questions (1) and (2) we know that the wavepacket is centered at x_0 as given in Eq. (2) in position, and at $p_0 = 0$ in momentum. The state $|0\rangle$ in turn is also a Gaussian wave packet centered at $x = 0$ and from question (9) we know that it has the same width as the state $|\lambda\rangle$ because the uncertainty in position is the same. Therefor, the state $|\lambda\rangle$ can be obtained from the state $|0\rangle$ by performing the translation $x \to x_0$. So U_λ is the finite translation operator. Recalling that the generator of translations is $\frac{i}{\hbar}p$ we get

$$U_\lambda = e^{-x_0 \frac{i}{\hbar} p}, \tag{38}$$

where x_0 is given by Eq. (2).

(11) From the previous question we know that $|\lambda$ is the ground state of a harmonic oscillator centered at x_0. Therefore, it can be obtained as a result of the measurement of the hamiltonian operator

$$H' = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 (x - x_0)^2. \tag{39}$$