Solution to the exam of QUANTUM PHYSICS I of 19 February 2025

(1) The probability of finding the system in one of the four states is the modulus squared of the coefficient in front of the state in $|\psi\rangle$:

$$P_{00} = |\langle 00|\phi\rangle|^2 = \frac{1}{6}|\langle 00|00\rangle|^2 = \frac{1}{6}; P_{01} = \frac{1}{3}; P_{10} = \frac{1}{3}; P_{11} = \frac{1}{6}. \tag{1}$$

(2)

$$P(q_1 = 0) = P_{01} + P_{00} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}. \tag{2}$$

(3)

$$P(q_2 = 0) = P_{10} + P_{00} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}. \tag{3}$$

(4) We are given the following operator:

$$A = i|10\rangle\langle 01| \tag{4}$$

Writing the states in vector form

$$|10\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}; |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}; |00\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}; |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}; \tag{5}$$

then the matrix form of A in this basis, using $A_{ij} = \langle i|A|j\rangle$ is

$$A = \begin{pmatrix}0 & i & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}. \tag{6}$$

Note we reordered the states, which we are always free to do, so that the matrices B and C below look nicer. where $|j\rangle = |10\rangle, |01\rangle, |00\rangle, |11\rangle$. The Hermitian conjugate (complex conjugate and transpose) of the matrix Eq. 6 is straightforwardly

$$A^\dagger = \begin{pmatrix}0 & 0 & 0 & 0\\-i & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}. \tag{7}$$

showing that A is not a Hermitian operator, and thus not a valid observable.

(5) We are given the operator $B = A + A^\dagger$, whose matrix representation is

$$B = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

To find out what are the possible measurement outcomes of this observable, we diagonalize the matrix to find its eigenvalues. You can focus on the little submatrix that contains the non-zero matrix elements:

$$det[\begin{matrix} -\lambda & -i \\ i & -\lambda \end{matrix}] = \lambda^2 - 1 = 0 \quad \longrightarrow \quad \lambda = \pm 1, \tag{9}$$

and we ignore the zero eigenvalues of B. The normalized eigenstates corresponding to these eigenvalues are

$$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + i|10\rangle); \\ |-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - i|10\rangle). \end{aligned} \tag{10}$$

Now we compute the probability of the two measurement outcomes:

$$\begin{aligned} P(\lambda = +1) = |\langle + |\psi_0\rangle|^2 = |\frac{1}{\sqrt{2}}(\langle 01| - i\langle 10|)(\frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle)|^2 = \frac{1}{6}|1-i|^2 = \frac{1}{3} \\ P(\lambda = -1) = |\langle - |\psi_0\rangle|^2 = |\frac{1}{\sqrt{2}}(\langle 01| + i\langle 10|)(\frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle)|^2 = \frac{1}{6}|1+i|^2 = \frac{1}{3}. \end{aligned} \tag{11}$$

where in the second equality we omitted the ket $|00\rangle$ and $|11\rangle$, because they are clearly ortogonal to $|\pm\rangle$. Note that these two probabilities only add up to 2/3 instead of 1, because we have not computed the one for the zero eigenvalue.

(6) p.37 of the book

(7) The easiest way to compute this probability is to just subtract the two probabilities in Eq. 11 from 1, because the probs have to add up to one in the end. This gives a probability of 1/3.

The state that the system collapses to after measurement is the projection of $|\psi_0\rangle$ onto the subspace with 0 eigenvalue, namely the state

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle). \tag{12}$$

(8) The question is whether B is compatible with a freshly defined operator C. We will check if they commute:

$$\begin{aligned} [C, B] &= [|10\rangle\langle 01| + |01\rangle\langle 10|, i|10\rangle\langle 01| - i|01\rangle\langle 10|] \\ &= -2i|10\rangle\langle 10| + 2i|01\rangle\langle 01| = 2i(|01\rangle\langle 01| - |10\rangle\langle 10|), \end{aligned} \tag{13}$$

where non-orthogonal products have been put to zero and orthogonal ones to one. Clearly this is nonzero, so B and C are not compatible. This means they do not share a set of eigenfunctions and cannot be measured simultaneously.

We are also asked to compute explicitly the uncertainty product for a generic state, which you can do by putting the commutator into the Heisenberg uncertainty principle:

$$\sigma_B \sigma_C \geq \frac{1}{2}|\langle [C,B] \rangle| = \langle \psi | (|01\rangle\langle 01| - |10\rangle\langle 10|) | \psi \rangle. \tag{14}$$

(9) If you measure B at $t = 0$, the state will collapse to either $|+\rangle$ or $|-\rangle$ (Eq. 10). So we have to first time evolve these two states using the given Hamiltonian, and then compute the probability that the state is again the same at some later time t using the time-evolved states.

$$\begin{aligned} |\phi(0)\rangle &= |\pm\rangle \\ \rightarrow |\phi(t)\rangle = U(t)|\phi(0)\rangle = e^{-iHt/\hbar}|\phi(0)\rangle &= e^{-i\lambda C t}|\pm\rangle. \end{aligned} \tag{15}$$

Since the time-evolution operator depends on the operator C, we have to determine how C acts on the eigenstates of B. We will first determine the eigenstates and -values of C and then write the eigenstates of B i.t.o. the eigenstates of C, which gives us how C acts on the eigenstates of B. To avoid confusion, we will call C's eigenstates $|+\rangle_C, |-\rangle_C$ and B's eigenstates $|+\rangle_B, |-\rangle_B$.

C has eigenvalues ± 1 with eigenstates

$$\begin{aligned} |+\rangle_C &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle); \\ |-\rangle_C &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \end{aligned} \tag{16}$$

Now we write the eigenstates of B (Eq. 10) i.t.o. Eq. 16:

$$\begin{aligned} |+\rangle_B &= \frac{1}{2}[(1+i)|+\rangle_C + (1-i)|-\rangle_C]; \\ |-\rangle_B &= \frac{1}{2}[(1-i)|+\rangle_C + (1+i)|-\rangle_C]. \end{aligned} \tag{17}$$

Now we can easily determine the time evolution, because we know how C acts on its own eigenstates:

$$e^{-i\lambda C t}|\pm\rangle_C = e^{\mp i\lambda t}|\pm\rangle_C. \tag{18}$$

So for the B-states:

$$\begin{aligned} |+(t)\rangle_B = e^{-i\lambda C t}|+\rangle_B &= \frac{1}{2}\left[(1+i)e^{-i\lambda t}|+\rangle_C \ + \ (1-i)e^{i\lambda t}|-\rangle_C\right]; \\ |-(t)\rangle_B = e^{-i\lambda C t}|-\rangle_B &= \frac{1}{2}\left[(1-i)e^{-i\lambda t}|+\rangle_C \ + \ (1+i)e^{i\lambda t}|-\rangle_C\right]. \end{aligned} \tag{19}$$

Now we can compute the probabilities:

$$
\begin{aligned}
P_{|+\rangle_B} &= |_B\langle +| + (t)\rangle_B|^2 \\
&= |\frac{1}{2}[(1-i)\langle +|_C + (1+i)\langle -|_C]\frac{1}{2}[(1+i)e^{-i\lambda t}|+\rangle_C + (1-i)e^{i\lambda t}|-\rangle_C]|^2 \\
&= \frac{1}{16}|2e^{-i\lambda t} + 2e^{i\lambda t}|^2 = \frac{1}{4}|e^{-i\lambda t} + e^{i\lambda t}|^2 \\
&= \frac{1}{4}|2\cos\lambda t|^2 = \cos^2\lambda t.
\end{aligned} \tag{20}
$$

For $|-\rangle_B$ the calculation and result are exactly the same, so also $P_{|-\rangle_B} = \cos^2\lambda t$.

(10) We are asked for some state that saturates the uncertainty principle (i.e. a state that satisfies Eq. 14 with an equal sign). A necessary and sufficient condition for this is that

$$
B|\psi\rangle = i\lambda C|\psi\rangle, \tag{21}
$$

where λ is a real number.

As ansatz we take the general superposition

$$
|\psi\rangle = \alpha|00\rangle + \beta|10\rangle + \gamma|01\rangle + \delta|\psi\rangle. \tag{22}
$$

Now we compute both sides of Eq. 21 to determine possible values of the coefficients and λ:

$$
\begin{aligned}
B|\psi\rangle &= -i\beta|01\rangle + i\gamma|10\rangle; \\
i\lambda C|\psi\rangle &= i\lambda(\beta|01\rangle + \gamma|10\rangle) = i\lambda\beta|01\rangle + i\lambda\gamma|10\rangle.
\end{aligned} \tag{23}
$$

This leads to the pair of equations

$$
\begin{aligned}
\beta &= -\lambda\beta; \\
\gamma &= \lambda\gamma
\end{aligned} \tag{24}
$$

so we find either $\lambda = -1$, with $\gamma = 0$ and β fixed by normalization, or $\lambda - 1$, with $\beta = 0$ and γ fixed by normalization

Picking the first solution we get the state

$$
|\psi\rangle = |01\rangle. \tag{25}
$$

We can check that this state saturates Eq. 14:

$$
\langle 01|[C,B]|01\rangle = \frac{1}{2}|2i\langle 01|(|01\rangle\langle 01| - |10\langle\rangle 10|)|01\rangle| = |2i|/2 = 1 \tag{26}
$$

and

$$
\begin{aligned}
\sigma_B &= \sqrt{\langle B^2\rangle - \langle B\rangle^2}; \\
\langle B^2\rangle &= \langle 01|(-|10\rangle\langle 10| + |01\rangle\langle 01|)|01\rangle = 1; \\
\langle B\rangle^2 &= 0; \\
\to \sigma_B &= 1.
\end{aligned} \tag{27}
$$

A similar calculation also yields $\sigma_C = 1$, so in total we can conclude that

$$\sigma_B \sigma_C = \frac{1}{2} |\langle [B, C] \rangle| = 1, \tag{28}$$

So indeed our state saturates the uncertainty principle.

The calculation for the other solution is similar.

Note that while we are only asked to give one state, you can add whatever superpositions of $|11\rangle$ and $|00\rangle$ to this, because these do not contribute anything to the uncertainty product. For example, you can do $|\psi\rangle = \frac{1}{\sqrt{3}}(|11\rangle + |00\rangle + |10\rangle)$.

NB: Equivalent results can be obtained by putting the operators in matrix form, i.e. using the states in the form of Eq. 5.

(11) Take an arbitrary state

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle. \tag{29}$$

To fully describe this state, we need to know all coefficients. In total, this corresponds to 8 degrees of freedom, since they are complex numbers. However two of these are a normalization and a phase, hence we need six independent operators. Now we nore that the operators B and C are the Pauli matrices σ_1 and σ_2 in the subspace of states $|10\rangle$ and $|01\rangle$. This fixes completely one qubit in this subspace. If we also introduce the same Pauli matrices in the orthogonal subspace this fixes the other qubit:

$$B' = i(|00\langle\rangle 11| - |11\rangle\langle 00|) \tag{30}$$

$$C' = |00\langle\rangle 11| + |11\rangle\langle 00|. \tag{31}$$

Finally, we need a pair of operators in order to fix the relative normalization and phase of the two qubits. These could be for instance

$$B'' = i(|00\rangle\langle 10| - |10\rangle\langle 00|) \tag{32}$$

$$C'' = |00\rangle\langle 10| + |10\rangle\langle 00|. \tag{33}$$