PROVA IN ITINERE DI FISICA QUANTISTICA

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Traccia di soluzione

(1) Upon performing an energy measurement the wave function of the system collapses into one of the two energy eigenstates $|1\rangle$ and $|2\rangle$ of which the state $|\psi\rangle$ is a superposition. The possible outcomes of the measurements are respectively E_1 with probability $\frac{1}{3}$ and E_1 with probability $\frac{2}{3}$, with

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{8ma^2}. \tag{1}$$

Upon a momentum measurement, the wave function collapse into a momentum eigenstate. Each energy eigenstate is the superposition of two momentum eigenstates with values

$$p_n = \pm\hbar\frac{n\pi}{2a}, \tag{2}$$

where the probability of finding each of the eigenvalues (positive or negative) upon performing a measurement of the momentum are equal. Hence the possible outcomes of the momentum measurements will be $\pm p_1$ each with probability $\frac{1}{6}$ and $\pm p_2$ each with probability $\frac{2}{6}$, with p_i given by Eq. (2)

(2) Recall that in the position basis the wave function is given by

$$\langle x|n\rangle = \begin{cases} \frac{1}{\sqrt{a}}\sin(k_n x) \text{ for even } n \\ \frac{1}{\sqrt{a}}\cos(k_n x) \text{ for odd } n \end{cases}, \tag{3}$$

where

$$k_n = \frac{n\pi}{2a}. \tag{4}$$

The expectation value of the position is

$$\begin{aligned} \langle\psi|x|\psi\rangle &= \frac{1}{2}\langle 1|x|1\rangle + \frac{2}{3}\langle 2|x|2\rangle + i\frac{\sqrt{2}}{3}\langle 1|x|2\rangle - i\frac{\sqrt{2}}{3}\langle 2|x|1\rangle, && (5)\\ &= \frac{1}{2}\langle 1|x|1\rangle + \frac{2}{3}\langle 2|x|2\rangle = 0 && (6) \end{aligned}$$

where in the first step we made use of the fact that $\langle 1|x|2\rangle = \langle 2|x|1\rangle$. The two terms in (6) vanish because they are integrals over x of x times sine or cosine squared, so they are the integral of an odd function on an even domain.

The expectation value of the momentum is

$$\langle\psi|p|\psi\rangle = \frac{1}{3}\left(\langle 1|p|1\rangle + 2\langle 2|p|2\rangle + i\sqrt{2}\langle 1|p|2\rangle - i\sqrt{2}\langle 2|p|1\rangle\right). \tag{7}$$

In the coordinate basis the momentum operator acts as $\langle x|p|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\psi(x)$. Remembering (3) we see that the diagonal terms vanish because they are again the integral of an odd function on an even domain. The off-diagonal terms are

$$\langle 1|p|2\rangle = \int_{-a}^{a} \frac{1}{\sqrt{a}}\cos\left(\frac{\pi x}{2a}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\frac{1}{\sqrt{a}}\sin\left(\frac{\pi x}{a}\right) = -i\frac{4\hbar}{3a} \tag{8}$$

where after taking the derivative the integral is given on the exam sheet; note that $\langle 1|p|2\rangle = \langle 2|p|1\rangle^*$. Using this, and plugging it into (7), we find

$$\langle\psi|p|\psi\rangle = \frac{8\sqrt{2}\hbar}{9a}. \tag{9}$$

The calculation of the expectation value of the energy is straightforward. Given that $|\psi\rangle$ is a superposition of energy eigenstates with eigenvalues (1), we find

$$\langle\psi|H_0|\psi\rangle = \frac{1}{3}E_1 + \frac{2}{3}E_2 = \frac{1}{3}(1+2*4)\frac{\pi^2\hbar^2}{8ma^2} = \frac{3\pi^2\hbar^2}{8ma^2}. \tag{10}$$

(3) The time evolution of the state $|\psi\rangle$ as governed by the Hamiltonian H_0 can be explicitly written as

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}H_0 t}|\psi\rangle = \frac{1}{\sqrt{3}}e^{-\frac{i}{\hbar}E_1 t}|1\rangle + i\sqrt{\frac{2}{3}}e^{-\frac{i}{\hbar}E_2 t}|2\rangle. \tag{11}$$

The time-dependent expectation value of the position is $\langle\psi(t)|x|\psi(t)\rangle$, where the diagonal terms vanish as a result of (11) and (6). By explicitly writing the off-diagonal terms, one finds

$$\begin{aligned}\langle\psi(t)|x|\psi(t)\rangle &= i\frac{\sqrt{2}}{3}e^{-\frac{i}{\hbar}(E_2-E_1)t}\langle 1|x|2\rangle - i\frac{\sqrt{2}}{3}e^{\frac{i}{\hbar}(E_2-E_1)t}\langle 1|x|2\rangle & (12)\\ &= \frac{2\sqrt{2}}{3}\sin\left((E_2-E_1)t/\hbar\right)\langle 1|x|2\rangle & (13)\\ &= \frac{2\sqrt{2}}{3}\sin(\omega t)\langle 1|x|2\rangle, & (14)\end{aligned}$$

where we again used $\langle 1|x|2\rangle = \langle 2|x|1\rangle^*$, $\langle 1|x|2\rangle$ can be calculated using (3) and one of the integrals given on the exam sheet, and

$$\hbar\omega = E_2 - E_1 = \frac{3}{8}\frac{\hbar^2\pi^2}{ma^2}. \tag{15}$$

We have $\langle 1|x|2\rangle = \frac{32a}{9\pi^2}$, which we can plug into (14), resulting in

$$\langle\psi(t)|x|\psi(t)\rangle = \frac{64\sqrt{2}}{27\pi^2}a\sin(\omega t). \tag{16}$$

The expectation value of position depends on time. This is a consequence of the fact that the Hamiltonian does not commute with the position operator, $[H,\hat{x}] \neq 0$, hence the position is not conserved.

The expectation value of the energy is not time dependent, and hence equal to what we found before in (10). This is a consequence of the fact that the Hamiltonian describing the time evolution of the system is time-independent: the system is invariant upon time translations and thus energy is conserved.

(4) We note that the potential is given by

$$\langle x|V(\hat{x})|x'\rangle = V(x)\delta(x-x'), \tag{17}$$

but clearly

$$V_1(x) = V_0(x-a) \tag{18}$$

(for example $V_1(2a) = V_0(a)$). So

$$V_1(x)\delta(x-x') = \langle x|V_1(\hat{x})|x'\rangle = V_0(x-a)\delta(x-x') = \langle x-a|V_0(\hat{x})|x'-a\rangle = \langle x|T_a^{-1}V_0(\hat{x})T_a|x'\rangle \tag{19}$$

(see Eq. (4.56) of the textbook). Therefore the potential $V_1(\hat{x})$ can be obtained from $V_0(\hat{x})$ by the translation

$$H_1 = T_a^{-1} H_0 T_a. \tag{20}$$

where the translation operator is

$$T_\delta = e^{\frac{i}{\hbar}\delta\hat{p}}. \tag{21}$$

Next we can use the unitarity of T to find

$$T_a H_1 T_a^{-1}|n\rangle = H_0|n\rangle = E_n|n\rangle \quad \Rightarrow \quad H_1 T_a^{-1}|n\rangle = E_n T_a^{-1}|n\rangle. \tag{22}$$

It follows that the operators H_0 and H_1 are unitarily equivalent, they have the same eigenvalues, and their eigenvectors are related by

$$|n^{(1)}\rangle = T_a^{-1}|n^{(0)}\rangle \tag{23}$$

where by $|n^{(i)}\rangle$ we denote the eigenvectors of H_i:

$$H_i|n^{(i)}\rangle = E_n|n^{(i)}\rangle, \qquad i = 1,\ 2. \tag{24}$$

Hence in the coordinate basis we have

$$\psi_n^{(1)}(x) = \langle x|n^{(1)}\rangle = \langle x|T_a^{-1}|n^{(0)}\rangle = \psi_n^{(0)}(x-a). \tag{25}$$

(5) Call $|\chi\rangle$ the new state in which we have to calculate expectation values. Equation (23) implies that it is found from the state $|\psi\rangle$ Eq. (3) of the assigment using

$$|\chi\rangle = T_a^{-1}|\psi\rangle. \tag{26}$$

Equation (21) immediately implies that $[p, T_a] = 0$, from which it follows that

$$\langle\chi|\hat{p}|\chi\rangle = \langle\psi|T_a\hat{p}T_a^{-1}|\psi\rangle = \langle\psi|\hat{p}|\psi\rangle, \tag{27}$$

so the expectation value of the momentum is unchanged, and it is still given by Eq. (9).

From (24) we know that the energy eigenvalues are unchanged as a result of the translation. This means that the expectation value of the energy is also unchanged, and it is still given Eq. (10). More formally

$$\langle\chi|H_1|\chi\rangle = \langle\psi|T_a H_1 T_a^{-1}|\psi\rangle = \langle\psi|H_0|\psi\rangle. \tag{28}$$

Finally, the expectation value of x is

$$\langle\chi|\hat{x}|\chi\rangle = \langle\psi|T_a\hat{x}T_a^{-1}|\psi\rangle = \langle\psi|\hat{x}+a|\psi\rangle = a, \tag{29}$$

where we have used the action of the translation on the operator $\hat{x}$ (Eq. (4.80) of the textbook) and, in the last step, Eq. (6).