

Solution to the exam of QUANTUM PHYSICS I of 16 September 2026

(1) Heisenberg equations for x and p

$$\dot{x} = \frac{1}{i\hbar}[x, H] = \frac{1}{2mi\hbar}[x, p^2] = \boxed{\frac{p}{m}}, \quad (1)$$

$$\dot{p} = \frac{1}{i\hbar}[p, -\kappa x] = \frac{-\kappa}{i\hbar}(-i\hbar) = \boxed{\kappa}. \quad (2)$$

(2) Heisenberg equations for T and V

With $T = p^2/2m$ and $V = -\kappa x$, only the cross commutators survive:

$$\dot{T} = \frac{1}{i\hbar}[T, V] = \frac{-\kappa}{2mi\hbar}[p^2, x] = \frac{-\kappa}{2mi\hbar}(-2i\hbar p) = \boxed{\frac{\kappa p}{m}}, \quad (3)$$

$$\dot{V} = \frac{1}{i\hbar}[V, T] = -\dot{T} = \boxed{-\frac{\kappa p}{m}}. \quad (4)$$

(3) Time dependence of T and V

Integrating (1) (the equations are linear, so the operator solution is the classical one):

$$p(t) = p_0 + \kappa t, \quad x(t) = x_0 + \frac{p_0 t}{m} + \frac{\kappa t^2}{2m}, \quad (5)$$

where $x_0 \equiv x(0)$, $p_0 \equiv p(0)$. Hence

$$\boxed{T(t) = \frac{(p_0 + \kappa t)^2}{2m} = T(0) + \frac{\kappa p_0 t}{m} + \frac{\kappa^2 t^2}{2m}}, \quad (6)$$

$$\boxed{V(t) = -\kappa x(t) = V(0) - \frac{\kappa p_0 t}{m} - \frac{\kappa^2 t^2}{2m}}. \quad (7)$$

(4) $\langle p|H|p'\rangle$ in the momentum basis

In this basis $\langle p|p'\rangle = \delta(p - p')$ and $\langle p|x|p'\rangle = i\hbar \frac{d}{dp}\delta(p - p')$. Therefore

$$\langle p|H|p'\rangle = \frac{p'^2}{2m} \langle p|p'\rangle - \kappa \langle p|x|p'\rangle = \boxed{\frac{p^2}{2m} \delta(p - p') - i\kappa\hbar \frac{d}{dp} \delta(p - p')}. \quad (8)$$

(5) $\langle p|H|p'\rangle$ in the position basis

Insert two completeness relations and use $\langle x|p\rangle = e^{ipx/\hbar}/\sqrt{2\pi\hbar}$:

$$\langle p|H|p'\rangle = \int dx dx' \langle p|x\rangle \langle x|H|x'\rangle \langle x'|p'\rangle, \quad (9)$$

$$= \int dx dx' \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \kappa x \right] \delta(x-x') \frac{e^{ip'x'/\hbar}}{\sqrt{2\pi\hbar}}. \quad (10)$$

For the first term we perform the x' integral using the Dirac delta. In the second term we use $x e^{i(p'-p)x/\hbar} = i\hbar \frac{d}{dp} e^{i(p'-p)x/\hbar}$. We get

$$\langle p|H|p'\rangle = \frac{\hbar}{4\pi m} \frac{p^2}{\hbar^2} \int dx e^{-i(p'-p)x/\hbar} - \frac{i\kappa}{2\pi} \frac{d}{dp} \int dx e^{i(p'-p)x/\hbar} \quad (11)$$

Integrating the two terms gives $\boxed{\frac{p^2}{2m} \delta(p-p') - i\kappa\hbar \frac{d}{dp} \delta(p-p')}$.

(6) Theory: the operator $\hat{x}$ in the momentum basis

See Eqs. (4.138)-(4.139) of the textbook.

(7) Eigenstates of H

Using the solution to question (4), the eigenvalue equation $H|\psi_E\rangle = E|\psi_E\rangle$ becomes a *first-order* ODE for $c_E(p) = \langle p|\psi_E\rangle$:

$$\frac{p^2}{2m} c_E - i\kappa\hbar \frac{dc_E}{dp} = E c_E \quad \implies \quad \frac{dc_E}{dp} = \frac{i}{\kappa\hbar} \left(E - \frac{p^2}{2m} \right) c_E. \quad (12)$$

Integrating,

$$\boxed{c_E(p) = N \exp \left[\frac{i}{\kappa\hbar} \left(E p - \frac{p^3}{6m} \right) \right]}, \quad (13)$$

where N (the boundary condition) is fixed by normalization. The spectrum is continuous and unbounded, $E \in (-\infty, \infty)$, and every eigenvalue is non-degenerate. Requiring and improper normalization condition $\langle \psi_E | \psi_{E'} \rangle = \delta(E - E')$ leads to

$$N = \frac{1}{\sqrt{2\pi\kappa\hbar}}. \quad (14)$$

(8) Energy measurement, then momentum measurement

After the energy measurement: The outcome E_0 projects the system onto the corresponding eigenstate, i.e. the $E = E_0$ case of (7):

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\kappa\hbar}} \exp \left[\frac{i}{\kappa\hbar} \left(E_0 p - \frac{p^3}{6m} \right) \right]. \quad (15)$$

Possible momentum outcomes: The spectrum of p is continuous, so any $p \in (-\infty, \infty)$ can be found.

Probability:

$$P(p) = \left| \tilde{\psi}(p) \right|^2 = \frac{1}{2\pi\kappa\hbar}, \quad (16)$$

which is constant, independent of p : the momentum is completely uncertain.

Post-measurement state: If the outcome is p_0 , the system collapses onto the momentum eigenstate

$$|p_0\rangle, \quad \psi_{p_0}(x) = \langle x|p_0\rangle = \frac{e^{ip_0x/\hbar}}{\sqrt{2\pi\hbar}}, \quad (17)$$

(9) $\langle x \rangle$ and minimum Δx at time t_1

From (3) the Heisenberg operators are

$$\hat{x}(t) = \hat{x} + \frac{t}{m} \hat{p} + c(t), \quad \hat{p}(t) = \hat{p} + \kappa t, \quad (18)$$

where $\hat{x} \equiv \hat{x}(0)$, $\hat{p} \equiv \hat{p}(0)$ and $c(t) = \frac{\kappa t^2}{2m}$. All expectation values below are taken in the state at $t = 0$, which does not evolve in this picture.

Mean value: Taking $\langle \cdot \rangle$,

$$\boxed{\langle \hat{x}(t_1) \rangle = x_0 + \frac{\langle \hat{p}_0 \rangle t_1}{m} + \frac{\kappa t_1^2}{2m}.} \quad (19)$$

Minimal uncertainty: Consider the commutator

$$[\hat{x}, \hat{x}(t_1)] = \frac{t_1}{m} [\hat{x}, \hat{p}] = \frac{i\hbar t_1}{m}. \quad (20)$$

Using the uncertainty relation $\Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|$, we get

$$\Delta x(0) \Delta x(t_1) \geq \frac{1}{2} \frac{\hbar t_1}{m}. \quad (21)$$

Inserting $\Delta x(0) = \sigma$ and dividing,

$$\boxed{\Delta x(t_1) \Big|_{\min} = \frac{\hbar t_1}{2m\sigma}, \quad \text{i.e.} \quad \Delta^2 x(t_1) \Big|_{\min} = \frac{\hbar^2 t_1^2}{4m^2 \sigma^2}.} \quad (22)$$

(10) Time-dependent $\kappa(t)$

Case $t_0 < 0$: Here $\kappa(t) = 0$ for every $t \geq 0$, so the evolution for $t > 0$ is the one of question (9) with $\kappa = 0$:

$$\hat{x}(t) = \hat{x} + \frac{t}{m} \hat{p}, \quad \hat{p}(t) = \hat{p}, \quad (23)$$

and therefore

$$\boxed{\langle \hat{x}(t_1) \rangle = \hat{x}_0 + \frac{\hat{p}_0 t_1}{m}.} \quad (24)$$

Since the term proportional to κ in Eq. (19) does not contain any operator the commutator Eq. (20) is unchanged, so the uncertainty relation gives

$$\boxed{\Delta x(t_1) \Big|_{\min} = \frac{\hbar t_1}{2m\sigma}.} \quad (25)$$

Case $0 < t_0 < t_1$: Repeating the computation of question (9), integrate the Heisenberg equations separately on the two intervals. On $0 < t < t_0$ the force is κ_0 and (29) applies, so at t_0

$$\hat{x}(t_0) = \hat{x} + \frac{t_0}{m} \hat{p} + \frac{\kappa_0 t_0^2}{2m}, \quad \hat{p}(t_0) = \hat{p} + \kappa_0 t_0. \quad (26)$$

On $t > t_0$ the force vanishes, so the motion is free starting from these values,

$$\hat{x}(t_1) = \hat{x}(t_0) + \frac{t_1 - t_0}{m} \hat{p}(t_0) = \hat{x} + \frac{t_1}{m} \hat{p} + \frac{\kappa_0 t_0}{m} \left(t_1 - \frac{t_0}{2} \right). \quad (27)$$

Taking the expectation value,

$$\boxed{\langle x(t_1) \rangle = x_0 + \frac{p_0 t_1}{m} + \frac{\kappa_0 t_0}{m} \left(t_1 - \frac{t_0}{2} \right).} \quad (28)$$

This again differs from Eq. (39) only by a number so the commutator is again unchanged, giving the same width:

$$\boxed{\Delta x(t_1) \Big|_{\min} = \frac{\hbar t_1}{2m\sigma}.} \quad (29)$$

(11) Measurement of $O = |\psi\rangle\langle\psi|$ and the limit $\sigma \rightarrow 0$

The measurement: Since O is a projector, its only possible eigenvalues are 1 and 0. The corresponding eigenspaces are read off from

$$O|\psi\rangle = |\psi\rangle \quad (\lambda_O = 1), \quad O|\varphi\rangle = 0 \quad \text{for every } |\varphi\rangle \text{ with } \langle\psi|\varphi\rangle = 0 \quad (\lambda_O = 0): \quad (30)$$

the eigenvalue 1 belongs to the one-dimensional ray spanned by $|\psi\rangle$, the eigenvalue 0 to its orthogonal complement.

Limit $\sigma \rightarrow 0$: We notice that $\int_{-\infty}^{\infty} dx \psi(x) = 1$ for all values of σ . It follows that

$$\lim_{\sigma \rightarrow 0} \psi(x) = \delta(x). \quad (31)$$

Hence in this limit the state $|\psi\rangle$ is a position eigenstate located at $x = 0$, and after the measurement the position uncertainty is $\Delta x(0) = 0$, so Eq. (21) implies

$$\boxed{\lim_{\sigma \rightarrow 0} \Delta x(t) = \infty \quad \text{for all } t > 0.} \quad (32)$$

Localizing the particle ever more sharply at $t = 0$ forces $\Delta p = \hbar/2\sigma$ to grow without bound, and the packet disperses immediately.