**QUANTUM MECHANICS II EXAM**

25 January 2022

*Answers sheet*

Consider a system whose dynamics is described by the Hamiltonian

$$H_0 = \frac{\kappa}{2I}(\vec{L})^2 + \hbar \vec{B} \cdot \vec{L} \tag{1}$$

where $\vec{L} = \vec{x} \times \vec{p}$ are the angular momentum operators, $I$ and $\kappa$ are positive real constants, and $\vec{B}$ is a three-dimensional vector with real components (not necessarily positive).

(1) In the case in which $\vec{B} = 0$ the Hamiltonian Eq. (1) becomes proportional to the operator $(\vec{L})^2$ and thus the eigenstates are the angular momentum eigenfunctions $\mid lm\rangle$ with integer $l$ satisfying $l \geq 0$ and $-l \leq m \leq l$. The eigenvalues are

$$E_l = \frac{\kappa}{2I}\hbar^2 l(l+1). \tag{2}$$

(2) Since the eigenvalues Eq. (2) do not depend on $m$, the degeneracy is $d = 2l + 1$.

(3) Let us, without loss of generality, assume that $\vec{B}$ points in the $z$ direction. In this case the eigenvalues of Eq. (1) are found to be

$$E_{lm} = \frac{\kappa}{2I}\hbar^2 l(l+1) + \hbar^2 m|\vec{B}|. \tag{3}$$

In this case, the eigenvalues Eq. (3) are not independent of $m$ and hence they are non-degenerate.

(4) The Heisenberg equations of motion of the angular momentum operator are:

$$\frac{\mathrm{d}L_j}{\mathrm{d}t} = \frac{i}{\hbar}[H, L_j] = \frac{i}{\hbar}[\hbar B_i L_i, L_j] \tag{4}$$

$$= -\hbar B_i \epsilon_{ijk} L_k = \hbar \epsilon_{jik} B_i L_k, \tag{5}$$

where we used the commutation relations of angular momentum $[L^2, L_j] = 0$ and $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$.

In vector form Eq. (5) can be written as

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \hbar \vec{B} \times \vec{L}. \tag{6}$$

(5) In the case where $\kappa = 0$ the Heisenberg equations of motion of the position operator are:

$$\frac{\mathrm{d}x_l}{\mathrm{d}t} = \frac{i}{\hbar}[H, x_l] = \frac{i}{\hbar}[\hbar B_i L_i, x_l] = i[B_i \epsilon_{ijk} x_j p_k, x_l] \tag{7}$$

$$= iB_i \epsilon_{ijk}(x_j [p_k, x_l] + [x_j, x_l] p_k) \tag{8}$$

$$= \hbar \epsilon_{ijl} B_i x_j, \tag{9}$$

where to get the third line we used that $[x_i, x_j] = 0$ and $[x_i, p_j] = i\hbar\delta_{ij}$.

In vector form this can be written as

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \hbar \vec{B} \times \vec{x}. \tag{10}$$

The case for the momentum operator can be found analogously:

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \hbar \vec{B} \times \vec{p}. \tag{11}$$

(6) See the complement 25 in the textbook.

(7) The correction of the first order perturbation at any energy level of the unperturbed Hamiltonian is

$$\Delta E_1 = \epsilon\hbar|\vec{A}|\langle l,m \mid L_x \mid l,m\rangle = \frac{1}{2}\epsilon\hbar|\vec{A}|\langle l,m \mid (L_+ + L_-) \mid l,m\rangle = 0, \tag{12}$$

where $L_\pm = L_x \pm iL_y$.

(8) Because $\vec{B} = 0$, we can diagonalize the Hamiltonian using the basis of eigenstates of $J^2$, $J_z$, $S^2$, and $L^2$:

$$H = H_0 + \hbar\vec{\sigma}\cdot\vec{L} = H_0 + 2\vec{S}\cdot\vec{L} = H_0 + (\vec{J}^2 - \vec{L}^2 - \vec{S}^2), \tag{13}$$

where $\vec{J} = \vec{L} + \vec{S}$ is the total angular momentum. The eigenvalues are

$$E = \frac{\kappa}{2I}\hbar^2 l(l+1) + \hbar^2(j(j+1) - l(l+1) - s(s+1)) \tag{14}$$

$$= \hbar^2\left[j(j+1) + \left(\frac{\kappa}{2I} - 1\right)l(l+1) - \frac{3}{4})\right]. \tag{15}$$

$$\tag{16}$$

If $l = 0$ then $j = \frac{1}{2}$, if instead $l \geq 1$ then $j = l - \frac{1}{2}$ or $j = l + \frac{1}{2}$. This however does not lead to degeneracy because of the factor multiplying $l(l+1)$. The only degeneracy is then related to possible values of $j_z$, namely $d = 2j + 1$.

(9) In this problem and the next we assume $|\vec{B}| < \frac{\kappa}{2I}$, so the ground state has $l = 0$. Without loss of generality, we assume that $\vec{B}$ points along the (positive) $z$ axis, so the first excited state has $l = 1$, $m = -1$. The correction of the second order perturbation to the energy of the first excited state of the unperturbed Hamiltonian is then

$$\Delta E_2 = \epsilon^2\hbar^2|\vec{A}|^2\frac{|\langle 1,0 \mid L_x \mid 1,-1\rangle|^2}{E_{1,-1} - E_{1,0}} \tag{17}$$

$$= \epsilon^2\hbar^2|\vec{A}|^2\frac{|\langle 1,0 \mid \frac{1}{2}(L_+ + L_-) \mid 1,-1\rangle|^2}{E_{1,-1} - E_{1,0}}, \tag{18}$$

$$= -\epsilon^2\hbar^2|\vec{A}|^2\frac{1}{2|\vec{B}|}, \tag{19}$$

where we used $L_- \mid 1,-1\rangle = 0$ and $L_+ \mid 1,-1\rangle = \hbar\sqrt{2} \mid 1,0\rangle$.

(10) At $t = 0$ the system is in a superposition of the first and second excited states of the Hamiltonian $H_0$:

$$\mid \psi_0\rangle = \frac{1}{2}\left(\mid 1,-1\rangle + \mid 1,0\rangle\right). \tag{20}$$

The probability that at a time $t$ the system is a superposition of the same two states orthogonal to $\mid \psi_0\rangle$, thus in

$$\mid \psi_1\rangle = \frac{1}{2}\left(\mid 1,-1\rangle - \mid 1,0\rangle\right), \tag{21}$$

is

$$P = \left| \langle \psi_1 \mid e^{\frac{i}{\hbar} t H_0} \mid \psi_0 \rangle \right|^2, \tag{22}$$

$$= \left| \langle \psi_1 \mid e^{-it\frac{\kappa}{I}\hbar} e^{-it|\vec{B}|L_z} \mid \psi_0 \rangle \right|^2, \tag{23}$$

$$= \left| \frac{1}{\sqrt{2}} \langle \psi_1 \mid \left( e^{i\hbar t|\vec{B}|} \mid 1, -1\rangle + \mid 1, 0\rangle \right) \right|^2, \tag{24}$$

$$= \left| \frac{1}{2} \left( e^{i\hbar t|\vec{B}|} - 1 \right) \right|^2, \tag{25}$$

$$= \left| \frac{1}{2} \left( e^{\frac{i}{2}\hbar t|\vec{B}|} - e^{-\frac{i}{2}\hbar t|\vec{B}|} \right) \right|^2, \tag{26}$$

$$= \sin^2 \left( \frac{1}{2}\hbar|\vec{B}|t \right), \tag{27}$$

(11) For the Hamiltonian

$$H = H_0 + \hbar\vec{\sigma} \cdot \vec{L} = \frac{\kappa}{2I}(\vec{L})^2 + \hbar\vec{B} \cdot \vec{L} + \hbar\vec{\sigma} \cdot \vec{L}, \tag{28}$$

the eigenvalue spectrum cannot be determined exactly because the with $\vec{B} \neq 0$ the eigenstates of the Hamiltonian are also eigenstates of $L_z$, and it is not possible to diagonalize $L_z$, $J^2$, $S^2$, and $L^2$ simultaneously. Thus we need to treat the term to $\vec{\sigma}$ as a perturbation. The first order correction is

$$\Delta E = \langle l, m, s, m_s \mid \hbar\vec{\sigma} \cdot \vec{L} \mid l, m, s, m_s \rangle, \tag{29}$$

$$= \langle l, m, s, m_s \mid \hbar \left( \sigma_x L_x + \sigma_y L_y + \sigma_z L_z \right) \mid l, m, s, m_s \rangle, \tag{30}$$

$$= \langle l, m, s, m_s \mid 2S_z L_z \mid l, m, s, m_s \rangle, \tag{31}$$

$$= 2\hbar^2 m_s m. \tag{32}$$