Solution to the exam of QUANTUM PHYSICS II of 21 February 2025

(1) Using the following coordinate decomposition

$$
\begin{aligned}
\vec{x}_1 &= \vec{R} + \vec{r}/2; \\
\vec{x}_2 &= \vec{R} - \vec{r}/2; \\
\vec{p}_1 &= \vec{P}/2 + \vec{p}; \\
\vec{p}_2 &= \vec{P}/2 - \vec{p},
\end{aligned}
\tag{1}
$$

we can write the Hamiltonian as

$$
\begin{aligned}
H &= H_{COM} + H_{rel}; \\
H_{COM} &= \frac{\vec{P}^2}{2M}; \\
H_{rel} &= \frac{\vec{p}^2}{2\mu} - \frac{e^2}{|\vec{r}|},
\end{aligned}
\tag{2}
$$

where $M = 2m$ and $\mu = m/2$

Looking at the relative part, this is just the Hydrogen atom We already know the energy spectrum of Hydrogen, and we can add the energy of the free particle that is the COM part:

$$
E_n = -\frac{\mu e^4}{2\hbar} \frac{1}{n^2} + \frac{\vec{P}^2}{2M}.
\tag{3}
$$

(2) Again we treat the relative (Hydrogen atom) and COM (free particle) parts separately. The Hydrogen G.S. wavefunction is $\psi_{100}^{Hy}(\vec{r}) = \frac{e^{-r/a}}{\sqrt{\pi a^3}}$, with $a = \hbar^2/\mu e^2$ in this case, and $r = |\vec{r}|$. Since H_{COM} is the Hamiltonian of a free particle with mass M, the eigenfunctions are just plane waves $\psi_{COM}(\vec{R}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{R}}$, with $\vec{k} = \vec{P}/\hbar$ the COM wavevector and in the G.S. $\vec{k} = 0$. In total we then have for the GS:

$$
\psi_{GS} = \frac{1}{(2\pi)^{3/2}} \, \psi_{100}^{Hy}(\vec{r}).
\tag{4}
$$

(3) Here we will ignore the free particle WF, because it will cancel out in the modulus.

$$\langle r \rangle = \int d^3\vec{r}\,|\psi_{100}(r)|^2\,r$$

$$= \int d\cos\theta\,d\phi\,dr\,r^2|\psi_{100}(r)|^2 r$$

$$= 4\pi \int_0^\infty dr\,r^3|\psi_{100}(r)|^2 \tag{5}$$

$$= \frac{4}{a^3} \int_0^\infty dr\,r^3 e^{-2r/a}$$

$$= \frac{4}{a^3}\frac{6}{(2/a)^4} = 3a/2 = 3/(me^2),$$

where in the last line we used the given equation to solve the integral.

(4) We have a probability density

$$P(r) = |\psi_{100}(r)|^2 d^3\vec{r}$$

$$= 4\pi r^2\,|\psi_{100}(r)|^2\,dr \tag{6}$$

$$= \frac{4\pi r^2}{\pi a^3}e^{-2r/a} = \frac{4r^2}{a^3}e^{-2r/a}.$$

We maximize $P(r)$ by setting $dP(r)/dr = 0$:

$$\frac{dP}{dr} = \frac{8r}{a^3}e^{-2r/a}(1 - \frac{r}{a}) = 0, \tag{7}$$

which leads to $r_{max} = 0$ or $\pm a$. Since r is a distance, $r = a = \frac{2}{me^2}$ is the nontrivial solution.

(5) $n = 2$ and $l \neq 0$, so $l = 1$. Then $m = -1, 0, 1$. The most general state we can construct is $|2, 1, m\rangle$, so we write a superposition of the different states $|2, 1, m\rangle$:

$$|\psi\rangle = a|2, 1, 1\rangle + b|2, 1, 0\rangle + c|2, 1, -1\rangle, \tag{8}$$

where the coefficients satisfy $|a|^2 + |b|^2 = |c|^2 = 1$.

(6) See Eq. (10.135)-(10.141) of the textbook.

(7)
$$P = |211\rangle\langle 21-1| + |21-1\rangle\langle 211|. \tag{9}$$

In the basis $|211\rangle, |210\rangle, |21-1\rangle$, we can put this operator in matrix form:

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{10}$$

Note that this is the first Pauli matrix in 2D if you remove the non-acting state $|210\rangle$. So we have to determine the eigenvalues and eigenstates of the Pauli matrix (where we take $|211\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|21-1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$):

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{11}$$

We already know the eigenvalues of this matrix, namely $-1, 1$, with corresponding eigenvectors

$$
\begin{aligned}
|\psi_+\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|211\rangle + |21-1\rangle); \\
|\psi_-\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|211\rangle - |21-1\rangle).
\end{aligned}
\tag{12}
$$

Next to this, we have the eigenvalue 0 corresponding to the non-acting state $|210\rangle$. Looking at the general state Eq. 8 this has probability $|b|^2$. The other probabilities are given by

$$
\begin{aligned}
P(\lambda = 1) &= \frac{1}{2}|(a^*\langle 211| + b^*\langle 210| + c^*\langle 21-1|)(|211\rangle + |21-1\rangle)|^2 = \frac{|a+c|^2}{2}; \\
P(\lambda = -1) &= \frac{1}{2}|(a^*\langle 211| + b^*\langle 210| + c^*\langle 21-1|)(|211\rangle - |21-1\rangle)|^2 = \frac{|a-c|^2}{2}.
\end{aligned}
\tag{13}
$$

You see that all probabilities nicely add up to $|a|^2 + |b|^2 + |c|^2 = 1$.

So the possible measurement outcomes are -1 and 1, with the state collapsing to $\frac{1}{\sqrt{2}}(|211\rangle - |21-1\rangle)$ and $\frac{1}{\sqrt{2}}(|211\rangle + |21-1\rangle)$ respectively. The corresponding probabilities are given by Eq. 13.

(8) $H_p = \epsilon\vec{B} \cdot \vec{L} = \epsilon(B_x L^x + B_y L^y + B_z L^z)$. We can choose $\vec{B}$ to be along the z-axis, so that $H_p = \epsilon B_z L^z$. We know that

$$L^z|nlm\rangle = \hbar m |nlm\rangle, \tag{14}$$

so the three degenerate states are already eigenvectors of the perturbation:

$$E_m^{(1)} = \epsilon B_z \hbar m. \tag{15}$$

The respective eigenvalues are the first-order correction to the energy.

This perturbation lifts the degeneracy between $|21-1\rangle, |210\rangle$ and $|211\rangle$, because it introduces an explicit m-dependence in the energy.

(9) $\lambda = 1$ with $|\psi_+(0)\rangle = \frac{1}{\sqrt{2}}(|211\rangle + |21-1\rangle)$ the state after measurement.

At time t

$$|\psi_+(t)\rangle = e^{iHt/\hbar}|\psi_+(0)\rangle = e^{i(H+H_p)t/\hbar}|\psi_+(0)\rangle, \tag{16}$$

$$= e^{-i\frac{E_n}{\hbar}}e^{iH_pt/\hbar}|\psi_+(0)\rangle. \tag{17}$$

The hydrogen energies E_n energies cancel out in the modulus of the probability because they are an overall factor. The other exponent containing $H_p = \epsilon B_z L_z$ acts different on the eigenstates of m, so we have to compute

$$|\psi_+(t)\rangle = \frac{1}{\sqrt{2}}e^{itE_n/\hbar}(e^{i\epsilon B_z t/\hbar}|211\rangle + e^{-i\epsilon B_z t/\hbar}|21-1\rangle) \tag{18}$$

Then the probability of this state evolving to the $\lambda = -1$ state is given by

$$P_{\lambda \to -1} = |\langle \psi_-(0)|\psi_+(t)\rangle|^2 = |\frac{1}{2}(\langle 211| - \langle 21-1|)(e^{i\epsilon B_z t/\hbar}|211\rangle + e^{-i\epsilon B_z t/\hbar}|21-1\rangle)|^2$$
$$= |\frac{1}{2}(e^{i\epsilon B_z t/\hbar} - e^{-i\epsilon B_z t/\hbar})|^2 = \sin^2 \frac{\epsilon B_z t}{\hbar}. \tag{19}$$

(10) To prove the result we note that
$$[L_z, z] = [L_z, z^2] = 0 \tag{20}$$

This implies that

$$0 = \langle nlm|[L_z, z^2]|nlm'\rangle = (m - m')\langle nlm|z^2|nlm'\rangle. \tag{21}$$

This equation can only be satisfied if either $m = m'$, or $\langle nlm|z^2|nlm'\rangle = 0$, i.e. only the diagonal matrix elements are nonzero.

(11) The perturbation is given by

$$V(t) = Ez^2 e^{-t/\tau}\Theta(t), \tag{22}$$

and is therefore dependent on the z^2 operator. The transition probability from the one state to the other is given by the following in time-dependent perturbation theory:

$$P(-1 \to 1) = Ee^{-t/\tau}\Theta(t)\langle\psi_-|z^2|\psi_+\rangle \propto ((\langle 211| - \langle 21-1|)z^2(|211\rangle + |21-1\rangle)). \tag{23}$$

Using the result from the previous question, we can simplify this to

$$P(-1 \to 1) \propto \langle 211|z^2|211\rangle - \langle 21-1|z^2|21-1\rangle. \tag{24}$$

But we now note that $P|211\rangle = |21-1\rangle$, where P is the parity operator. It follows that $\langle 211|z^2|211\rangle = \langle 21-1|z^2|21-1\rangle$ so the transition probability is zero.

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