Solution to the exam of QUANTUM PHYSICS II of 17 June 2025

(1) We need to determine the energy spectrum of $H_i = \frac{\vec{p}^2}{2m} - \frac{3e^2}{r_i}$. This is a regular Coulomb potential with a Bohr radius given by $a = \frac{\hbar^2}{3me^2}$ and so we know the energy spectrum

$$E_{n_i} = -\frac{m(3e^2)^2}{2\hbar^2 n_i^2}. \tag{1}$$

The degeneracy is $D = 2n_i^2$ if there is spin, and $D = n_i^2$ if there is no spin.

(2) We deal with distinguishable particles, so there is no Pauli exclusion principle and the GS is just the state with all particles in the lowest orbital $n = 1$. For distinguishable particles, we can add the separate energies and the state is the product of all three states:

$$E_n = \sum_i E_{n_i} = -\frac{m(3e^2)^2}{2\hbar^2}\left(\frac{1}{n_1^2} + \frac{1}{n_2^2} + \frac{1}{n_3^2}\right)$$
$$|\psi\rangle = |n_1 l_1 m_1\rangle|n_2 l_2 m_2\rangle|n_3 l_3 m_3\rangle \tag{2}$$

(3) Here we do include the d.o.f.'s of the spin, but we do not use the exclusion principle because the particles are still distinguishable.
GS: $n_1 = n_2 = n_3 = 1$, $E_1 = -\frac{3m(3e^2)^2}{2\hbar^2}$ with D=8, since we have no spatial degeneracy, but each particle has D=2 from the spin.
1st excited state: one of the particles goes to the second orbital, so for the distinguishable particles we then have three possible states:

$$|\psi\rangle = |100\rangle_1|100\rangle_2|2lm\rangle_3 \otimes \chi \tag{3}$$
$$|\psi\rangle = |100\rangle_1|2lm\rangle_2|100\rangle_3 \otimes \chi \tag{4}$$
$$|\psi\rangle = |2lm\rangle_1|100\rangle_2|10\rangle_3 \otimes \chi. \tag{5}$$

Furthermore for the $|2lm\rangle$ state in each case we have four possibilities, namely $|100\rangle$, $|21-1\rangle$, $|210\rangle$, $|211\rangle$. Therefore for each of the three above states we have $D_{spat} = 1 \cdot 1 \cdot 4 = 4$, $D_\chi = 8$, and there is the degeneracy of having three states with the same energy, so $D_{tot} = 4 \cdot 8 \cdot 3 = 96$.

(4) For identical particles we need to antisymmetrize the total wave function, which implies the Pauli exclusion principle. Hence in the GS we have two particles with $n = 1$ and one with $n = 2$, and the GS energy is

$$E_1 = -\left(1 + 1 + \frac{1}{4}\right)\frac{m(3e^2)^2}{2\hbar^2} = -\frac{9}{4}\frac{m(3e^2)^2}{2\hbar^2}. \tag{6}$$

The wave function is the completely antisymmetric combination of three one-particle wave functions $|nlm\chi\rangle$ (with $\chi = \pm$ indicating the value of the third componento of spin): $|100+\rangle$, $|100-\rangle$, $|2lm\chi\rangle$. As seen in the previous question the state $|2lm\chi\rangle$ is four times degenerate since $l \leq n$ and $-l \leq m \leq l$, and moreover in each case χ can take the two values $\pm$. Hence in total $D = 4 \cdot 2 = 8$.

(5) Since the fundamental state is a combination of $n = 1$, $n = 1$ and $n = 2$, we have a combination of $l = 0$, $l = 0$ and $l = 0, 1$. So in total, $l_{tot} = 0, 1$. Since $L^2 f = \hbar^2 l(l + 1)f$, we have $L^2 = 0$ or $L^2 = 2\hbar^2$ in the GS.

(6) See Sect. 10.3.1 of the textbook, Eqs. (10.42)–(10.46)

(7) The first particle is in the state $|2lm\rangle$, we assume no spin. Once the measurement of $L_z = +\hbar$ is done, the state is therefore

$$|\phi\rangle = |211\rangle. \tag{7}$$

Now we determine the possible measurement outcomes of $P_z = i\hbar\left(|l0\rangle\langle l1| - |l1\rangle\langle l0|\right)$. The outcome of the measurement of an operator is one of its egienstates. But the given operator has nonvanishing matrix elements only between the states $|\ell, 1\rangle$ and $|\ell, 0\rangle$. On the other hand the state $\phi\rangle$ has $\ell = 1$ so only the two states

$$|1\rangle = |l = 1, m = 1\rangle; \qquad |0\rangle = |l = 1, m = 0\rangle \tag{8}$$

are relevant because on any other state either the operator P_z has vanishing matrix element, or the state $|\phi\rangle$ has vanishing scalar product. Therefore we want to diagonalize P_z in this subspace, where it is equal to

$$P_z = \hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \hbar\sigma_y. \tag{9}$$

Since this is just the second Pauli matrix, we already know its eigenvalues and -vectors, namely $\pm\hbar$ with states

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}(|1\rangle + i|0\rangle) \tag{10}$$

$$|-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}(|1\rangle - i|0\rangle) \tag{11}$$

Now we can write $|\phi\rangle$ in terms of the P_z eigenstates as $|\phi\rangle = |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$. So the measurement of P_z can result in:
* $+\hbar$ with probability $|\langle +|\phi\rangle|^2 = 1/2$, where the state collapses to $|+\rangle$
* $-\hbar$ with probability $|\langle -|\phi\rangle|^2 = 1/2$, where the state collapses to $|-\rangle$

(8) $|\phi\rangle = |211\rangle$ and $H' = H_1 + H_z = H_1 + BP_z$ with $B > 0$. We can calculate the time evolution operator as follows:

$$|\phi(t)\rangle = e^{-iE_2 t/\hbar} e^{-iBP_z t/\hbar} |\phi(0)\rangle = e^{-iE_2 t/\hbar} e^{-iBP_z t/\hbar} |1\rangle, \tag{12}$$

where E_2 is the regular $n = 2$ one-particle Coulomb energy as previously calculated, and we are only interested in the action of the operator P_z in the subspace of states Eq. (8), for the same reason as in question (7). What is left is to calculate the H_z exponent:

$$e^{-iBP_zt/\hbar} = e^{-iB\sigma_y t} = \cos(Bt)\cdot I^{2\times 2} - i\sin(Bt)\cdot\sigma_y. \tag{13}$$

After applying this to $|\phi(0)\rangle = |1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, we find for the time-evolved state:

$$|\phi(t)\rangle = e^{-iE_2t/\hbar}(\cos Bt|1\rangle + \sin Bt|0\rangle). \tag{14}$$

The probability that the state is the same as at t=0 is then

$$|\langle\phi(t)|\phi(0)\rangle|^2 = |e^{-iE_2t/\hbar}(\cos Bt\langle 1| + \sin Bt\langle 0|)|211\rangle|^2 = \cos^2 Bt. \tag{15}$$

Note that E_2 does not contribute to the probability because P_z commutes with H_1 and therefore the eigenvalue of H_1 is conserved.

(9) We are asked to consider the same Hamiltonian as in the previous question, but now regard H_z as a perturbation. Since the unperturbed Hamiltonian has a degeneracy of 4 at $n = 2$, we use degenerate perturbation theory and diagonalize the perturbation. The four states are $|100\rangle$, $|21-1\rangle$, $|210\rangle$, $|211\rangle$. The matrix of P_z is

$$P_z = \hbar\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{pmatrix}. \tag{16}$$

As we know from problem (7) P_z only mixes the $|l = 1, m = 0\rangle$ and $|l = 1, m = 1\rangle$ states, and we know that its eigenstates in this subspace are the states $|+\rangle$ and $|-\rangle$ Eq. (10,11) with eigenvalues $\pm B\hbar$. For the states $|l = 0, m = 0\rangle, |1, -1\rangle$, the energy perturbation is zero.

It follows that the perturbed states are $|l = 0, m = 0\rangle, |l = 1, m = -1\rangle$ (or any linear combination thereof) with energy E_2 and degeneracy $D = 2$, and $|+\rangle$ and $|-\rangle$ with energy $E_2 \pm B\hbar$ respectively, nondegenerate.

Finally, we note that, as already mentioned, P_z commutes with the Hamiltonian H_1, and thus it can be diagonalized simultaneously. It follows that the exact spectrum of eigenstates and eigenvalyes of H' is the same as that of H for all states with the exception of the two states $|210\rangle$, $|211\rangle$, that are replaced by the states $|\pm\rangle$, with energies $E_2 \pm B\hbar$. This means that the first order perturbative result is actually exact.

(10) At first the system is in the spatial state with two particles in n=1 and one particle in n=2. The measurement of the angular momentum yields $L^z_{tot} = +\hbar m_{tot} = +\hbar$, so the state must be the two particles in the GS and the one in the n=2 state having $|l = 1, m = 1\rangle$. Because then indeed $L^z_{tot} = \hbar \cdot 1 + 0 + 0 = \hbar$.

Then S^z_{tot} is measured. The two particles in the spatial GS remain the same (one having spin up and one spin down, according to the exclusion principle). The particle in the first excited state has spin up or spin down after the measurement. The wave function of the entire system can then be written as

$$|n,l,m,s_z\rangle_{tot} = \frac{1}{\sqrt{6}}\Big(|100+\rangle_1|100-\rangle_2|111+\rangle_3 - |100+\rangle_1|111+\rangle_2|100-\rangle_3$$
$$-|100-\rangle_1|100+\rangle_2|111+\rangle_3 + |100-\rangle_1|111+\rangle_2|100+\rangle_3 +$$
$$|111+\rangle_1|100+\rangle_2|100-\rangle_3 - |111+\rangle_1|100-\rangle_2|100+\rangle_3\Big) \quad (17)$$

if the measurement gives $S_z^{tot} = 1/2$, and as

$$|n,l,m,s_z\rangle_{tot} = \frac{1}{\sqrt{6}}\Big(|100+\rangle_1|100-\rangle_2|111-\rangle_3 - |100+\rangle_1|111-\rangle_2|100-\rangle_3$$
$$-|100-\rangle_1|100+\rangle_2|111-\rangle_3 + |100-\rangle_1|111-\rangle_2|100+\rangle_3 +$$
$$|111-\rangle_1|100+\rangle_2|100-\rangle_3 - |111-\rangle_1|100-\rangle_2|100+\rangle_3\Big) \quad (18)$$

if the measurement gives $S_z^{tot} = -1/2$.

(11) $S_{tot}^z = m_s\hbar = \langle s_1^z\rangle + \langle s_2^z\rangle + \langle s_3^z\rangle$. Since the particles are identical, we can write

$$\langle s_i^z\rangle = m_s\hbar/3 = S_z^{tot}/3 = \pm\hbar/2, \quad (19)$$

where the last equality holds for $S_z^{tot} = \pm 1/2$. You can also calculate the mean values of the particle spins by looking at the wavefunctions. E.g. for particle 1 we have

$$\langle s_z^1\rangle = \frac{1}{6}\left(\frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2}\right) = \pm\frac{1}{6}. \quad (20)$$

Where the $\pm$ is for $S_z^{tot} = \pm 1/2$. The same is true for the other two particles, which makes sense because they are identical. The total mean value of the spin is then indeed $\langle s_z^1\rangle + \langle s_z^2\rangle + \langle s_z^3\rangle = \pm 1/2$.