Solution to the exam of QUANTUM PHYSICS II of 19 September 2025

$$H = H_1 + H_2, \tag{1}$$

where

$$H_1 = \frac{p_1^{x\,2} + p_1^{y\,2}}{2m} + \frac{p_2^{x\,2} + p_2^{y\,2}}{2m} + m\omega^2\left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right] \tag{2}$$

and

$$H_2 = \frac{p_1^{z\,2}}{2m} + \frac{p_2^{z\,2}}{2m} + m\omega'^2\left[(z_1 - z_2)^2\right] + Ez_1 + Ez_2. \tag{3}$$

(1) We are asked to split these Hamiltonians up into a center-of-mass (COM) part and a relative part. In order to do this, we define the following COM and relative coordinates:

$$q_x = \frac{x_1 + x_2}{2},\ q_y = \frac{y_1 + y_2}{2},\ q_z = \frac{z_1 + z_2}{2}, \tag{1}$$

$$r_x = x_1 - x_2,\ r_y = y_1 - y_2,\ r_z = z_1 - z_2, \tag{2}$$

$$P_x = p_1^x + p_2^x,\ P_y = p_1^y + p_2^y,\ P_z = p_1^z + p_2^z, \tag{3}$$

$$p_x = \frac{p_1^x - p_2^x}{2},\ p_y = \frac{p_1^y - p_2^y}{2},\ \frac{p_1^z - p_2^z}{2}. \tag{4}$$

We also define the total and reduced mass, respectively $M = 2m, \mu = m/2$. In terms of these coordinates, the Hamiltonians are written as

$$H_1^b = \frac{P_x^2 + P_y^2}{2M}, \quad H_1^r = \frac{p_x^2 + p_y^2}{2\mu} + m\omega^2(r_x^2 + r_y^2) = \frac{p_x^2 + p_y^2}{2\mu} + \frac{1}{2}m\Omega^2(r_x^2 + r_y^2), \tag{5}$$

$$H_2^b = \frac{P_z^2}{2M} + 2Eq_z, \quad H_2^r = \frac{p_z^2}{2\mu} + \frac{1}{2}\mu\Omega' 2 r_z^2, \tag{6}$$

where we have defined

$$\Omega = 2\omega \tag{7}$$

$$\Omega' = 2\omega'. \tag{8}$$

(2) The time evolution in the Heisenberg picture can be computed by using

$$\frac{dO_H}{dt} = \frac{i}{\hbar}[H, O_H]. \tag{9}$$

H_i^r commutes with the COM position and momentum, so we only use the COM Hamiltonian $H^b = \frac{P_x^2+P_y^2+P_z^2}{2M} + 2Eq_z$ to compute the time evolution.

The equations of motion for the three position operators q_i are

$$\frac{dq_j}{dt} = \frac{i}{\hbar}[H^b, q_j] = \frac{i}{\hbar}[\sum_{j=1}^{3}\frac{P_j^2}{2M}, q_j] = \frac{i}{2\hbar M}[P_j^2, q_j]$$

$$= \frac{i}{2\hbar M}\left(P_j[P_j, q_j] + [P_j, q_j]P_j\right) = \frac{i}{2\hbar M}(2i\hbar P_j) = \frac{P_j}{M}. \quad (10)$$

The equations of motion for the momentum operators P^i differ according to whether $i = 1, 2$ or $i = 3$. For $i = 1, 2$ we get

$$\frac{dP^i}{dt} = \frac{i}{\hbar}[H, P^i] = 0, \quad (11)$$

while for $i = 3$ we get

$$\frac{dP_z}{dt} = \frac{i}{\hbar}[H, P_z] = \frac{2iE}{\hbar}(i\hbar) = -2E \quad (12)$$

Integrating the equations of motion for the momentum operators, we get

$$P_x(t) = P_x(0) \quad (13)$$

$$P_y(t) = P_x y0) \quad (14)$$

$$P_z(t) = P_z(0) - 2Et. \quad (15)$$

Substituting in the equations for the position operators, we get

$$q_x(t) = q_x(0) + \frac{P_x(0)}{m}t \quad (16)$$

$$q_y(t) = q_y(0) + \frac{P_y(0)}{m}t \quad (17)$$

$$q_z(t) = q_z(0) + \frac{P_z(0)}{m}t - Et^2. \quad (18)$$

(3) The total relative Hamiltonian is the sum of three one-dimensional harmonic oscillator (HO) Hamiltonians:

$$H^r = \sum_{j=x,y}\left(\frac{p_j^2}{2\mu} + \frac{1}{2}\mu\Omega_j^2 r_j^2\right) + \left(\frac{p_z^2}{2\mu} + \frac{1}{2}\mu\Omega_z^2 r_z^2\right). \quad (19)$$

The energy spectrum is

$$E = E_x + E_y + E_z = \hbar\Omega(n_x + \frac{1}{2}) + \hbar\Omega(n_y + \frac{1}{2}) + \hbar\Omega'(n_z + \frac{1}{2})$$

$$= \hbar\left(\Omega N + \Omega' n_z + \frac{3}{2}\right), \quad (20)$$

$$(21)$$

where $N = n_x + n_y$. Because $\Omega \neq \Omega'$ and they are said to be incommensurable, this is the sum of a one-dimensional and a two-dimensional isotropic HO. The degeneracy is therefore the same as that of the isotropic two-dimensional oscillator, and equal to the number of ways one can choose n_x and n_y with fixed Nm hence

$$d = N + 1. \tag{22}$$

(4) This is just a product of the harmonic oscillator ground states:

$$\psi(\vec{r}) = \left(\frac{M}{\pi\hbar}\right)^{3/4} \Omega^{1/2}\Omega'^{1/4} \exp -\frac{M}{2\hbar}\left[\left(\Omega(x^2 + y^2\right) + \Omega'^2 z^2\right]. \tag{23}$$

(5) We can write a general state of H_1^r as $|\psi\rangle = |n_x, n_y\rangle$. The first excited state has one of the components in the first excited state, and can be written as

$$|\psi_1\rangle = \alpha|1, 0\rangle + \beta|0, 1\rangle, \tag{24}$$

with the normalisation condition $|\alpha|^2 + |\beta|^2 = 1$. Since α and β both have a real and an imaginary component, we start with 4 independent parameters. Then we have -1 from the normalisation constraint and -1 because the overall phase is unobservable, so we are left with 2 independent parameters.

(6) See Sect. 11.2, Eqs. (11.39 -11.42) of the textbook.

(7) We need to write the operator $\mathcal{O}$ in the two-dimensional subspace spanned by the states $|\psi_1\rangle$ for general α and β. We can do this in matrix form, defining $|1, 0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|0, 1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We note that $\mu\Omega = m\omega$, hence the operators a_i, $a_i^\dagger$ are the usual creation and annihilation operators. The matrix of the operator O in the subspace then is

$$\mathcal{O} = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \lambda\sigma_1. \tag{25}$$

It has eigenvalues $\pm\lambda$ with corresponding eigenvectors

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{26}$$

The probabilities of measuring either λ or $-\lambda$ can then be calculated as

$$P(+\lambda) = |\langle +|\psi_1\rangle|^2 = \frac{1}{2}|(\langle 1, 0| + \langle 0, 1|)(\alpha|1, 0\rangle + \beta|0, 1\rangle)|^2 = \frac{|\alpha + \beta|^2}{2}, \tag{27}$$

$$P(-\lambda) = \frac{|\alpha - \beta|^2}{2}. \tag{28}$$

You can check that the probabilities nicely add up to one, as demanded by the normalisation constraint.

(8) The first excited state energy of the nonperturbed Hamiltonian is

$$E_1^{(0)} = \hbar\omega(n_x + n_y + 1) = 2\hbar\omega, \tag{29}$$

which has a degeneracy of 2 due to the fact that either $n_x, n_y = 1, 0$ or $n_x, n_y = 0, 1$. We must consequently use degenerate perturbation theory. The matrix of the perturbation is

$$H' = \epsilon\mathcal{O} = \epsilon\lambda\sigma_1. \tag{30}$$

The eigenstates and eigenvalues have been calculated in the previous exercise. The effect of the perturbation is to split the degenerate state into these eigenstates, with energies given by the sum of the unperturbed energy E_1 and these eigenvalues, i.e.

$$E_1^{\pm} = E_1 \pm \epsilon\lambda. \tag{31}$$

(9) We have the operator

$$j_a = \frac{1}{2}\sum_{i,j=1}^{2} a_i^\dagger \sigma_{ij}^a a_j = \frac{1}{2}(a_1^\dagger, a_2^\dagger)\sigma^a \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{32}$$

Now we calculate the commutation relations

$$\begin{aligned}
[j^a, j^b] &= \frac{1}{4}\sum_{ijkl}[a_i^\dagger\sigma_{ij}^a a_j, a_k^\dagger\sigma_{kl}^b a_l] = \frac{1}{4}\sum_{ijkl}\sigma_{ij}^a\sigma_{kl}^b[a_i^\dagger a_j, a_k^\dagger a_l] \\
&= \frac{1}{4}\sum_{ijkl}\sigma_{ij}^a\sigma_{kl}^b\left(a_i^\dagger[a_j, a_k^\dagger]a_l - a_k^\dagger[a_l, a_i^\dagger a_j]\right) \\
&= \frac{1}{4}\sum_{ijkl}\sigma_{ij}^a\sigma_{kl}^b\left(\delta_{jk}a_i^\dagger a_l - \delta_{li}a_k^\dagger a_j\right) \\
&= \frac{1}{4}\sum_{ijl}\sigma_{ij}^a\sigma_{jl}^b a_i^\dagger a_l - \sum_{ijk}\sigma_{ki}^b\sigma_{ij}^a a_k^\dagger a_j \\
&= \frac{1}{4}\sum_{ijk}[\sigma_{ij}^a, \sigma_{jk}^b]a_i^\dagger a_k = \frac{1}{2}\sum_{ijk} i\epsilon_{abc}\sigma_{kj}^c a_i^\dagger a_k & (33)\\
&= i\epsilon_{abc}j^c. & (34)
\end{aligned}$$

Therefore the operators j_a have the commutation relations of angular momentum operators (up to a factor $\hbar$). It follows that the spectrum of J^2 is the same as that of the total angular momentum, namely $j(j+1)$, with j integer or half-integer.

(10) For the normal ladder operators we have the following commutation relations:

$$[a_i, a_j^\dagger] = \delta_{ij}, \ \ [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0. \tag{35}$$

Calculating these relations for $A_\pm$ is a matter of plugging in the commutation relations for $a, a^\dagger$:

$$[A_\pm, A_\pm] = [A_\pm^\dagger, A_\pm^\dagger] = \frac{1}{2}(0+0+0+0) = 0, \tag{36}$$

$$[A_\pm, A_\pm^\dagger] = \frac{1}{2}(1+1+0+0) = 1, \tag{37}$$

$$[A_+, A_-^\dagger] = \frac{1}{2}(1-1) = 0. \tag{38}$$

So we can conclude that indeed these commutation relations are the same as for $a, a^\dagger$, namely:

$$[A_\rho, A_\sigma^\dagger] = \delta_{\rho\sigma}, \ [A_\rho, A_\sigma] = [A_\rho^\dagger, A_\sigma^\dagger] = 0, \tag{39}$$

with $\rho, \sigma \in +, -$.

(11) We first note that

$$H_1^r = a_x^\dagger a_x + a_y^\dagger a_y + 1 \tag{40}$$

$$= A_+^\dagger A_+ + A_-^\dagger A_- 1 = N_+ + N_- + 1, \tag{41}$$

where

$$N_\pm = A_\pm^\dagger A_\pm \tag{42}$$

are the usual number operators, with spectrum given by non-negative integers $n_\pm$. We then express O in terms of $A_\pm, A_\pm^\dagger$ by rewriting $a_x = \frac{A_+ + A_-}{\sqrt{2}}, \ a_y = \frac{A_+ - A_-}{\sqrt{2}}$:

$$O = (a_x^\dagger a_y + a_y^\dagger a_x) = A_+^\dagger A_+ - A_-^\dagger A_- = (N_+ - N_-). \tag{43}$$

It follows that the Hamiltonian can be written as

$$H_1^r + \epsilon\lambda O = \hbar\Omega(N_+ + N_- + 1) + \epsilon\lambda(N_+ - N_-). \tag{44}$$

The spectrum is thus

$$E_{n_+ n_-} = (\hbar\Omega + \epsilon\lambda)n_+ + (\hbar\Omega - \epsilon\lambda)n_- + \hbar\Omega. \tag{45}$$